Winning With Math

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Welcome to Winning With Math!

This program covers three main topics:

1. Voting Theory: Whenever a group of people come together to make a decision, they combine their individual preferences to make that decision. In politics, this happens any time there's an election with multiple candidates. But this also happens when friends decide what to eat, or when a business chooses who to hire. Our first chapter on voting theory studies the processes for making these decisions.
2. Apportionment: Apportionment studies situations where resources need to be divided between a group of people. In politics, this happens when using a state's size to decide how many representatives that state should have in Congress. This also happens any time you have resources that you don't want to break, but need to divide between a group of people (as a silly example, sharing five apples between three people).
3. Gerrymandering: Gerrymandering has been a frequent topic in the news and many major court cases. In the U.S., each state gets broken up into several political districts. Each district gets to elect a representative to Congress, and the methods used to break each state up into districts can swing the balance of power in Congress.

In each of these chapters, we'll look at several examples and case studies. We'll discuss different methods that can be used for making decisions and how mathematics helps us understand when a method is good or bad. We'll generally focus on examples from politics, where the choice of methods used can have a profound impact.

Throughout this program, our goal is to stretch your mathematical and critical reasoning skills. The math isn't too technical. Instead, the challenging part is going from the math and examples to the concepts and main ideas. In some sense, you can think of this program as an old-school pencil-sharpener for mathematical reasoning: it will hone those skills, but only through hard work and careful practice. Here is some advice:

- Learning new math—especially from a textbook and not in a classroom—is hard. Go through the program slowly and steadily: take breaks, reread sections, and expect to spend several minutes on each page.
- One of the best ways to learn math is to talk with others: try to find a group to work with, ask each other clarification questions, and work on the reading exercises together.
- If you get stuck, don't give up—go on to the next section, talk to someone, check reading exercises at the back of the newsletter, or come back later.

- We want this text to appeal to as broad of an audience as possible. It covers material at a range of difficulty, going all the way up to cutting-edge research ideas. We're excited to be able to show such advanced material.

This program has two types of practice questions to help you wrestle with the material:

1. Reading Exercises: short practice questions with answers at the end of each chapter. Try them as you work through the program, and if you get stuck, flip to the end and check your work.
2. Discussion Questions: more conceptual, reflective questions that help you engage with the material. These don't usually have easy answers. If any of the discussion questions seem interesting, please send us your responses and we'll try our best to write back.

Finally, feel free to write to us with any questions, comments, reflections, or discussion question answers. To respond, write "Attn: Math" on the cover of your envelope, and mail it back to Prisoner Express. That will get our attention!

CHAPTER 1: VOTING THEORY

As mentioned in the introduction, voting extends well beyond selecting a favorite candidate and choosing the winner with the most points. This method is only one of many ways we can make a choice from a pool of potential candidates. As we'll soon see, each way has benefits and drawbacks that can (and do) greatly impact the decisions we make.

Let's start with a small example to illustrate how different methods of voting could yield different outcomes. Suppose that a transportation company is trying to decide on a logo, and the design has been narrowed down to three options: Airplane, Boat, or Car. Everyone at the company lists their preferences in order, and the votes come in with the following results:

Ranked Voter Preferences	
5	A > B > C
1	A > C > B
3	B > A > C
1	C > A > B
1	B > C > A
6	C > B > A
total 17	

Figure 1

In the first row of the table in Figure 1, we can see that there are five voters whose top choice is an airplane (A), second choice is a boat (B), and third choice is a car (C). Likewise, the second row shows that only one voter listed the preference order of A as their first, C as their second, and B as their third choice. The other rows follow a similar pattern. Additionally, when we add the numbers in the first column (which correspond to how many people held each preference order) we get seventeen, the total number of voters.

Suppose you really believe that one particular choice would make for the best logo. If you were in charge of running the election between A, B, and C (and feeling particularly devious), you could use the voting data to make a very compelling case that any one of them "won." All it would take would be picking a reasonable voting method that produced your desired winner.

You could argue that C should win if you run the election the way most U.S. elections are: everyone gives one point to their favorite candidate. Then A gets six total points, B gets four, and C gets seven[1]. This is the method people tend to be most familiar with, which produces C as the winner.

[1] We will walk through these calculations in more detail later

However, if your goal is to see B win, you could use what is called the Anti-Plurality method, where everyone votes only for their least favorite choice. Then, whichever candidate has the fewest total downvotes wins. This would mean that A has seven points (from the one voter thought B>C>A and the six who thought C>B>A), B has two points, and C has eight. B has the best score (the fewest people who voted him down), so B wins by a safe margin!

Finally, let's say you think A is best. If held to a Plurality vote A and C are almost tied (six points vs seven) and B is the clear loser. You argue that the B voters' votes, rather than being "wasted," should go to the voters' second choices (A or C), in order to represent their opinions equally in the close competition between A and C. If this happens, then A gains three votes for a total of (6+3=) nine, and C only gains one, and therefore ends up with eight. This method, called Ranked Choice, finds A to be the winner.

RE 1) Based on the voting results in Figure 1, which logo do you think is the fairest choice?


A. Airplane


B. Boat


C. Car

Why should the answer you circled win, rather than either of the others?

The moral of this example is that different methods of voting—different methods of picking one "winner" from the preferences of a group—can lead to different outcomes. In other words, just by picking the voting method, you could game the system to *win with math*. Of course, we're not interested in gaming the system. But if three good methods all lead to three distinct outcomes, it begs the question: what do we do?

Voting Methods and Real-World Elections

In any situation where a group must collectively choose between three or more options, the way we account for individuals' preferences can drastically change the outcome. In the 2000 presidential election, George Bush won Florida with an incredibly slim 537 vote lead over Al Gore. Had Green Party candidate Ralph Nader not won over 97,000 of the six million votes cast in the state, expert consensus has it that Al Gore would have won in Florida and therefore the presidency. In 2002, Jacques Chiraq and Jean-Marie Le Pen advanced to the second round of France's presidential election after narrowly receiving the most votes of the sixteen total candidates—none had over 20%. The voting method produced one final round candidate who was a far-right radical (Le Pen), and one who was right-leaning moderate (Chiraq). As a result of the majority not wanting to elect a radical, Chiraq went on to win over 80% of the popular vote.

However, had the first round's third place candidate (left-of-center) advanced to the final round instead of Le Pen, the election would have been much tighter, and could've easily led to Chiraq losing. Here in the U.S., 2016 exit polls of the Super Tuesday primaries suggest that had the full rankings of voters' candidate preferences been considered, Donald Trump would have won only two of the eleven states, rather than seven. If this trend continued in the rest of the states, the Republican candidate in the 2016 election (and thus our current president) could have been completely different if the primaries had used a different voting method.

These three elections, and most elections with at least three candidates, have one thing in common: their outcomes were directly and heavily impacted not just by *who* people voted for, but by *how* these votes were counted. Should the candidate who has the most votes win, even if they are the worst choice in the eyes of everyone else who voted? What if another candidate who stands on more even ground is the second choice of a large majority, but as a result wins few "first choice" votes? Are some situations more fitting for selecting winners purely based on general approval and consensus that the candidate in question can do the job?

In this chapter, we will delve into voting theory by exploring the various means of choosing a single winner from a pool of three or more choices. We will discuss the key methods available to make these decisions, their strengths, weaknesses, and unintended consequences, along with the choices you have as a voter in reference to the different methods. The content in this chapter is as applicable to any communal decision-making situation as it is to politics, however. Throughout this chapter, we will continue to use our Airplane, Boat, Car logo example to provide a small-scale, non-political application to see firsthand how different voting methods produce different outcomes.

Plurality Method

In most elections in the United States (and in many places around the world), every voter casts one vote for their preferred candidate, and whoever receives the most votes wins. This method is simple; people have little trouble understanding how to read the ballots and cast their votes, and in general, ballots can be easily counted. If more than 50% of voters share the same favorite candidate, this method represents the majority's wishes and ensures that the majority-preferred candidate gets elected. One concern with Plurality, however, is it only takes a voters' top choice into consideration, and ignores any preferences they might have among the other candidates. This method, like all of the voting methods we'll learn about in this chapter, has strengths and weaknesses.

Let's revisit our logo example. To tally the plurality vote, we can ignore each voter's second and third choice candidate preferences, since a true Plurality election would only have voters list one candidate. The vote is tallied by counting the total number of voters who ranked a logo option first.

Figure 1

Ranked Voter Preferences	
5	A > B > C
1	A > C > B
3	B > A > C
1	C > A > B
1	B > C > A
6	C > B > A
total 17	

How we calculate votes for A:
As we can see in *Figure 1*, there are five voters who believe A > B > C, and there is one voter who thinks A > C > B. These are the two groups of voters who rank A as their top candidate, so we add the number of voters for each ordering to get 5 + 1 = **6**.

Check Your Understanding 1) Show how we calculate the plurality votes for B and C.

B:	**C**:
# Voters who think B > A > C? ______ # Voters who think B > C > A? ______ _____ + _____ = ______ total B votes	# Voters who think C > A > B? ______ # Voters who think C > B > A? ______ _____ + _____ = ______ total C votes

Answers:

Plurality Election Results	
Candidate	**Votes**
A	6
B	4
C	7

Figure 2

Under the Plurality method, the Car wins the logo contest with 7 points.

One weakness of the Plurality method is known as the *spoiler effect*, which occurs when two similar candidates divide a voter base of similar beliefs, but don't pull away votes from the third candidate, giving the third candidate an advantage. In the context of our logo contest, for example, voters may tend to fall into one of two categories: those who want a plain, ordinary form of transportation as the logo (Car), and those who want the logo to be a more 'exciting' form of transportation (Airplane or Boat). Even with only seventeen voters, we see the spoiler effect at play.

Referring to Figure 1, we see that five of the six total voters who preferred Airplane listed Boat as their second choice, while only one of them chose Car as the next-best candidate. Looking at those who voted for B as a first choice, three chose A as a second, while only one chose C. This indicates that the adventurous images of airplanes and boats are more closely aligned than the images of boats and cars, therefore splitting the voter base between listing Airplane as the top choice and Boat as the top choice.

We can reframe what this means more broadly. The airplane and boat logos are decidedly more adventurous than the car, and we see that more than half (6 As + 4 Bs = **10**) of the voters preferred one of the more adventurous logos. Suppose they all got together and agreed to vote A>B>C to make sure of getting an adventurous logo. Then A would have received 10 votes and won, rather than C. The spoiler effect occurs because voters do not generally strategize this way, and therefore split their votes across both candidates rather than vote as a unit.

As briefly mentioned in the introduction, the U.S. presidential election of 2000 came down to a hair-splittingly close tie between George W. Bush and Al Gore in Florida. In this instance, the spoiler effect diverted tens of thousands of votes from Al Gore to Ralph Nader, the Green Party candidate with somewhat similar ideologies as Gore. Polling data indicates that a majority of the Floridian voter base prefered Gore and Nader to Bush, but their support was divided between the two. Meanwhile, support for Bush was not diverted to other candidates to nearly the same extent. As a result, Bush beat Gore by a margin of just 537 votes, out of the millions cast in Florida. Had Nader withdrawn from the election seeing that he would not win, most political scientists generally agree that his withdrawal would favor Gore, and tip the balance in the other direction. Though we cannot be sure who Nader's 97,000 voters would have elected had he not been in the race, many estimates and exit poll results indicate that Gore would have been the likely winner.

RE 2) Three political candidates (we'll call them Alice, Bob, and Carl) are running against each other, and the winner is determined through the plurality method. Bob and Carl are in tight competition, and it looks likely that one of them will win. Alice is running on a platform somewhat similar to Bob, but doesn't have much support and will almost certainly not win. Let's say you somewhat prefer Alice to Bob, but greatly prefer both to Carl. Who would you vote for and why?

____

____

____

There is no right answer. As a voter, you have to decide: Do you vote sincerely for your preferred candidate, Alice? Or do you vote strategically, and cast your vote for Bob to help him beat Carl? One could argue that strategizing undermines the reason behind voting in the first place.

This brings us to one more issue with plurality voting, which can also be regarded through the lens of the spoiler effect. In our logo example, the car (C) won with seven of seventeen total votes, or 41%. Let's take a look at what happens if one candidate drops out. Suppose the logo-choosing committee decides that a boat wouldn't be an option for the logo after all, and they decide to remove it from the ballot. Now, everyone who originally voted for B must vote for their second choice candidate (as shown in Figure 3). Doing so shows us that A now gets nine votes, compared to C with only eight. Even though B did not win originally, removing it changes the winner from C to A!

Ranked Voter Preferences	
5	A > B > C
1	A > C > B
3	~~B~~ > (A) > C
1	C > A > B
1	~~B~~ > (C) > A
6	C > B > A
total 17	

Figure 3

Check Your Understanding 2) If A had dropped out, who would win? (Hint: Set this up in the same way as we did for when B is dropped.)

Ranked Voter Preferences	
5	A > B > C
1	A > C > B
3	B > A > C
1	C > A > B
1	B > C > A
6	C > B > A
total 17	

Figure 1

Calculate the winner if **A** drops out:

Answer: B wins! (To walk through the full solution, see the end of the chapter)

In other words, C only wins if both other options, A and B, are being considered at the same time! In the original plurality vote, C had the most votes, followed by A, and then B (C>A>B). If any candidate of the three is removed, however, the votes will produce winners in the following order: B>A>C (if A or C is removed, B wins, if B is removed, A wins).

Anti-Plurality Method

Instead of trying to choose the most favored candidate, we might try to select the least disliked candidate. The anti-plurality vote does this by assigning -1 point to each voter's least favorite choice, and zero to all the rest. Then, the candidate with the fewest votes against them wins. In our logo example, you could imagine a voter thinking "I could live with an airplane or boat as the logo... anything but a car!" They'd then vote against the car option by giving it -1 points.

Ranked Voter Preferences	
5	A > B > C
1	A > C > B
3	B > A > C
1	C > A > B
1	B > C > A
6	C > B > A
total 17	

Figure 1

<u>How we calculate Anti-Plurality votes for A</u>:
Based on our rankings, we can find the anti-plurality winner by giving A, B, and C -1 vote for each time they appeared in third place. As we can see from the *Figure 1*, there is one voter who believes B>C>A, and there are six voters who think C>B>A. These are the only two combinations where A is the bottom ranked candidate, so we add the number of -1 votes for each ordering to get 1 + 6 = **7** total negative points.

Check Your Understanding 2) Show how we calculate the Anti-Plurality votes for B and C.

B:	**C**:
# Voters who think A > C > B? ______ # Voters who think C > A > B? ______ ____ + ____ = ____ total negative B votes	# Voters who think A > B > C? ______ # Voters who think B > A > C? ______ ____ + ____ = ____ total negative C votes

Answers:

Anti-Plurality Election Results	
Candidate	Votes
A	-7
B	-2
C	-8

Figure 4

Using the Anti-Plurality method, the Boat wins with -2 points. The Car has the most downvotes out of all options, finishing with -8.

We can see that even though C got the most votes when using the plurality method, B was the least divisive, and therefore wins the anti-plurality because it has the fewest people who *really* don't approve.

The Anti-Plurality method has appeal because it prevents a majority from being unhappy with the outcome. While it may not select the "favorite" in the sense that people's top choice might not win, it minimizes the number of people seeing their last choice win. Just like with plurality, however, it is far from strategyproof. If voters have a preference for one candidate in particular, they have an incentive to vote against that candidate's strongest competitor. Regardless of whether or not they think their preference's closest competitor is the worst overall, the "strong" candidates are often the ones who garner the most negative votes, in some cases even leading a collectively agreed upon undesirable candidate to win.

RE 3) In a hypothetical three candidate election between Alice, Bob, and Carl, let's say Alice is the strongest, with Bob closely trailing. Most voters for both Alice and Bob can agree that Carl is the worst. Suppose Bob is your preferred candidate, Alice is acceptable but not great, and Carl is your least favorite by a wide margin. Who would you cast your anti-plurality vote for and why?

Carl seems like the obvious answer, but here's where voters tend to vote strategically, rather than honestly. Many of Bob's supporters know that by casting downvoting Alice (instead of their expressing their true distaste for Carl), they can level their competition and increase Bob's chances of winning. Conversely, Alice's supporters know that Bob's voters are likely to do so, and may cast their votes toward bringing down Bob rather than the collectively despised opponent Carl. As a result of these supporters being pinned against each other, Carl stands a reasonable chance of winning. Further, even if they did not win and everyone voted honestly (for the candidate they truly believed to be worst), then the discrepancy between Alice and Bob could come down to the minority of voters who believed one of them to be worse than Carl.

Borda Count

The Borda Count is one method that takes into consideration a voter's whole ordering of preferences, as opposed to just their first or last choice. Voters rank all the candidates in order. In a three-candidate election, each voter's top choice receives two points, second choice receives one point, and last choice receives zero. In general, for an election with more than three candidates, whoever is in last place gets zero points, the second to last gets one, the third to last gets two, and so on.

	Ranked Voter Preferences	
	5	A > B > C
	1	A > C > B
	3	B > A > C
	1	C > A > B
	1	B > C > A
	6	C > B > A
total	17	

Figure 1

How we calculate Borda Count votes for A:
As we can see, six voters list A as their first choice (five vote that A>B>C, and one votes that A>C>B). Four voters list A as their second choice (since three vote that B>A>C and one votes that C>A>B). Since Candidate A receives two points for every first place vote and one point for every second place vote, the total number of points A receives equals $(2 \times 6) + (1 \times 4)$ for a total of 16 points.

Check Your Understanding 4) Show how we calculate the Borda Count votes for B and C.

B:

\# Voters who rank B in first place? (B>A>C + B>C>A) ______ + ______ = ______

\# Voters who rank B in second place? (A>B>C + C>B>A) ______ + ______ = ______

2 × ______ + 1 × ______ = ______ total points for B

#first place #second place

C:

\# Voters who rank C in first place? (C>A>B + C>B>A) ______ + ______ = ______

\# Voters who rank C in second place? (A>C>B + B>C>A) ______ + ______ = ______

2 × ______ + 1 × ______ = ______ total points for C

#first place #second place

Answers:

Borda Count Election Results	
Candidate	**Points**
A	16
B	19
C	16

Figure 5

Under the Borda Count, the Boat wins with 19 points, and the Car finishes with 16 points.

The Borda Count has strong support for a variety of reasons. The spoiler effect is lessened by the fact if you like two similar candidates, you can support both without hurting either of their chances by splitting their votes. As a result, you also don't risk giving a relative advantage to third candidate with a different platform altogether. Further, the Borda Count is much more resilient to the withdrawal of a candidate. Unlike what we saw with Plurality, the ranking order cannot flip to the opposite outcome (Saari p.77). Especially because Plurality often disadvantages moderates, one of Borda Count's biggest advantages lies in how it tends to elect middle-of-the-road candidates that more than one party can support. In our case, this helps us choose a logo that most people are okay with.

Suppose a party, P1, runs four candidates and party P2 only runs one. Supporters for P1 will presumably rank the four candidates in some combination of the first four slots on the ballot (meaning that each will receive between one and four points). It is likely that they will list P2 as #5, and therefore award him or her no points. However, supporters for P2 only have one choice: They will list their one candidate at #1, but they cannot gain the relative advantage over P1 by also offering votes to the others they support. In cases like this where one party is running significantly more candidates than another, it is far more likely to win.

To illustrate this with an example, suppose that there are eight voters who support P1 and twelve who support P2. P1 runs candidates A, B, C with A as the clear preference. While B and C are weaker, all three share P1's affiliations. The second party, P2, only has one candidate running (D).

We can assume that the eight P1 voters will vote A>B>C>D (or with some similar combination of A, B, and C filling the first 3 slots). Let's say also that the twelve P2 voters vote that D>A>B>C:

A gets (8*3)+(12*2) = 48 votes
D gets 12*3 = 36 votes

Even though twelve voters listed D as a first choice, and only eight listed A, Candidate A (and therefore P1) wins by a landslide!

Despite the fact that the Borda Count is more strategy-proof against individual voters, it is still possible for parties to leverage the method to their advantage. As with the Anti-Plurality count, consider a voter who truly believes A>B>C but thinks B is more likely to win than C. That voter can still vote strategically and dishonestly to give B as few points as possible, voting first for A, second for C, and third for B. Using the Borda method, all a party has to do to gain a significant advantage over another is to run one strong candidate and several subpar candidates.

Ranked Choice

In recent years, Ranked Choice has gained traction in the U.S. for being easily implemented and more representative of a majority's wishes than Plurality. Also known as the Instant Run-off method, this method effectively combats the spoiler effect while maintaining a winner-by-plurality (and winner-by-majority) system.

How it works

1. Voters rank candidates in order of preference, and the first round works much like plurality: every voter's first choice is recorded, and gives their desired candidate +1 vote.
2. If no candidate after the initial round has a majority (>50%), whoever has the fewest votes is eliminated.
3. All voters who had listed the eliminated candidate as a first choice then have their vote given to their second choice in the next round.

By eliminating only the lowest performer at a time, votes can be reallocated in real time among *all* remaining candidates. As a result, one final candidate always wins by a 50% majority, and voters have a say in each successive elimination.

Ranked Choice Election Results				
Candidate	Votes (Round 1)	% Votes (Round 1)	Votes (Round 2)	% Votes (Round 2)
A	6	35%	9	53%
B	4	24%	0	0%
C	7	41%	8	47%

Figure 6

Taking a look at our hypothetical election, there can only be one supplementary voting round to reach a 50% majority, because eliminating the first candidate will produce a head-to-head matchup between the two remaining finalists. We can see in the results above that no candidate receives more than 50% of votes in the first round[2]. As a result, the candidate with the fewest votes (B) is eliminated, and B votes are redistributed to the second choice candidates of B's primary voters. For example, the three voters who said that B>A>C will have their votes recast to A, while the one voter with the preference B>C>A will now have their vote count toward C.

We see some surprising results in the second round tally. Since the majority of voters whose first choice was candidate B listed A as their second choice, their secondary support lifted A from second place in Round 1 into having majority support and winning the seat. While under the Plurality method C would have won, Ranked Choice produces A as the winner.

Ranked Choice also makes it safer to vote for a minority candidate (like Ralph Nader, for example). This is because unless there is already a majority for the third choice opponent, independent voters can count on their second choice counting where it matters, even if their first choice is quickly eliminated. Maine recently became the first statewide adopter of ranked choice for state elections; critics argue that the more complicated ballots depress voter turnout. Proponents of Ranked Choice, in contrast, argue that recent Ranked Choice elections have been implemented inexpensively and experience high turnout (Griffiths et al.). According to the Electoral Reform Society, another group that advocates for Ranked Choice voting, also argues that "Candidates are also incentivised to run less divisive campaigns, as candidates will want to become their opponent's voters second favourite candidate." It is important to note that this incentive also occurs for methods like the Borda Count, where voters give points to more than just one candidate.

[2] The first round tallies candidates ranked first in exactly the same way as the Plurality method. For an explanation of how these totals are reached, see the Plurality section.

RE 4) Suppose a generic Ranked Choice election has the following results:

Candidate	Votes (Round 1)	% Votes (Round 1)	Votes (Round 2)	% Votes (Round 2)
A	20	27%	0	0%
B	24	32%	34	45%
C	31	41%	41	55%

Figure 7

Your task is to determine how voters must have ranked each candidate relative to the others. Use your knowledge of how the Ranked Choice method tallies votes and eliminates candidates to fill out the Ranked Voter Preferences table in *Figure 8*. (Hint: What do the results from Round 1 tell us about voters' first choices? What does the reallocation of votes in Round 2 tell us about voters' second choices?)

Note: this is a pretty challenging exercise, and you can't do it perfectly. At the end, you'll be able to find the total number of voters who ranked B first, for example, but you won't be able to tell how many ranked B>C>A versus how many ranked B>A>C. The solution can be found at the end of the chapter.

Ranked Voter Preferences	
	A > B > C
	A > C > B
	B > C > A or B > A > C
	C > B > A or C > A > B
75	

Figure 8

Approval Voting

What if we just want to choose a logo that most people can agree with, regardless of how people stack their preferences against each other? Situations like these often benefit from using Approval voting, where the preference order is not as important as providing a winner most people see fit. In this method, voters cast votes for however many of the candidates they would be fine filling the role, and whichever candidate has the most support then wins. Seems simple, right? In principle, this method selects leadership (or logos) with the most supporters, without regard to how the candidates rank against each other. For our logo selection example, this method would likely select the most generally-popular option.

However, the problem with approval voting occurs with larger pools of candidates. In political elections, and even smaller elections for leadership in the workplace, school board, etc., what often happens is a surprising disadvantage to those with the most experience, who have likely built a more opinionated platform than potential underdogs they're running against. These qualified candidates frequently hold stronger and more divisive opinions, which in turn lowers the number of voters who would approve of their leadership. In an election with a handful of underdogs that generally seem likeable, approval voting can produce under-qualified candidates that most people like and vote for without necessarily thinking they're the best for the job. In approval voting, the ability to vote for as many candidates as one wants means that it doesn't cost people anything to express support for an underdog. As a result, approval voting could produce a weaker candidate than one with a stronger (and potentially more controversial) base.

Arrow's Impossibility Theorem and Monotonicity

By now we've explored five different voting methods, and have shown that without any change to voters' preferences, any of the logos in our example could be chosen fairly depending on what method is used. We're now faced with evaluating how fair these methods are relative to each other. Fortunately, there are some tools available to help us do just that. In the voting theory world, judgements about a methods' fairness are often made in reference to what are called voting criteria. These criteria capture the different ways in which a voting system might be fair or unfair; voting systems are then evaluated to see which criteria they pass and which they fail. A famous result, *Arrow's Impossibility Theorem*, tells us about how these different definitions of "fairness" are related. Provided that voters each provide a ranking of all candidates in an election (similar to A>B> C in our ongoing example), and that they are unrestricted in how they determine these rankings, the theorem tells us that for voting methods that produce a ranking of the candidates as an end result, only one type of voting method satisfies two reasonable ideas of fairness.

We've already begun to learn about some of the underlying concepts of these fairness criteria, which we call *Pareto efficiency* and *Independence of Irrelevant Alternatives (IIA)*. Pareto efficiency means if all voters prefer one candidate to another, then the candidate everyone prefers does better than the other. For example, if a method were to ignore the votes altogether and randomly pick a winner, it would not be Pareto because a candidate could win despite being unanimously less popular than another. For a voting method to be IIA, an outcome must be the same if an additional, non-winning candidate is introduced. In other words, if the spoiler effect can influence an election's outcome under a certain method, the method must not be IIA. We already saw when we learned about Plurality how easily IIA can be violated. In fact, because positional systems depend on how each candidate stacks up against non-winners, it is very difficult to find a method that satisfies the criterion: Plurality, Anti-Plurality, Ranked Choice, and Borda Count all violate it (and Approval Voting does not satisfy the initial conditions for comparison under Arrow's Theorem since it is not rank-based).

However, Borda Count, for example, does meet the conditions to satisfy the Pareto requirement. In any instance where all voters indicate that A>B>C, A will receive more points than B, and B will receive more points than C, therefore leading A to win.[3] The "impossibility" of Arrow's Theorem comes from the fact that in any election with three or more candidates, the only way for a voting method to satisfy both the IIA and Pareto efficiency fairness criteria is to be a dictatorship, where the outcome is representative of only one individual's preferences (Saari).

While both Pareto and IIA give valuable insight about a voting method's strengths and weaknesses, another metric called *monotonicity* is often used as an additional evaluator of fairness. Monotonicity says that if voters have ranked candidates and produced a certain outcome, then if a voter changes their preference order to rank a candidate higher than they had previously, that candidate cannot do worse after the preference order change. It seems like this would always be the case, right? Not quite. While Plurality, Anti-Plurality, and Borda Count all satisfy monotonicity, Ranked Choice does not.

Let's take another look at the Ranked Choice method, using modified results from our initial logo example. (Now instead of seventeen voters, we have 100). Here are our initial results shown below (*Figure 9.2*), produced from the Ranked Voter Preferences listed in *Figure 9.1* to its left.

Voter Preferences (modified)	
28	**A > B > C**
5	**A > C > B**
16	**B > A > C**
5	**C > A > B**
16	**B > C > A**
30	**C > B > A**
100	

Figure 9.1

Ranked Choice Election Results

Candidate	Votes (Round 1)	% Votes (Round 1)	Votes (Round 2)	% Votes (Round 2)
A	33	33%	49	49%
B	32	32%	0	0%
C	35	35%	51	51%

Figure 9.2

In this election we can see that C wins. What's interesting, however, is if we suppose that just two voters who ranked A>C>B wanted to show more support for candidate C (so their preference changes to C>A>B), who *already* won. Our new voter preferences table and results would look like this:

[3] For more information about Pareto efficiency, see *Chaotic Elections* by Donald Saari, referenced in the annotated bibliography

Voter Preferences (modified)	
28	A > B > C
3	A > C > B
16	B > A > C
7	C > A > B
16	B > C > A
30	C > B > A
100	

Figure 10.1

Ranked Choice Election Results				
Candidate	Votes (Round 1)	% Votes (Round 1)	Votes (Round 2)	% Votes (Round 2)
A	31	31%	0	0%
B	32	32%	60	60%
C	37	37%	40	40%

Figure 10.2

Just by ranking C over A rather than A over C, Candidate B (whose ranking did not change from last place in either of the two cases) *gained* support and ended up winning! In other words, *gaining* support made C lose.

Ranked Choice fails monotonicity, but this example is part of a larger message. These metrics to evaluate voting methods help us determine how well our elections at any scale hold up against recurring, predictable problems and ability to be gamed. Understanding the math behind these voting methods helps us predict winners and understand what factors lead to certain outcomes. But more importantly, developing such a mathematical basis can help us quantitatively evaluate how the methods we encounter daily shape real, impactful decisions that affect ourselves and those around us.

Summary

In this chapter we discussed a variety of voting methods including Plurality, Anti-Plurality, Ranked Choice, Borda Count and Approval Voting. Through mathematical examples, we examined the strengths and weaknesses of each of these methods, discussed how to minimize voters' and candidates' abilities to game the system, and learned how value judgments can help us decide what method may be most appropriate for a given election. Ultimately, we discovered how the seemingly minor choices we make in *how* votes are counted can have just as much influence over an election's outcome as the votes cast themselves. The content we covered in this chapter extends well beyond politics: groups must collectively make decisions all the time, and developing sound reasoning about how to make fairer, more representative choices is key to having your voice and the voices of those around you heard.

Discussion Questions

DQ 1) Now that we have been through a variety of voting methods for choosing one candidate among many... Which voting method would make you feel most heard? Are there any in particular where you would feel more motivated to vote than otherwise? Less?

DQ 2) Back to our logo contest, based on the voting methods we have discussed, which do you think is the fairest way to choose a winner? Would your choice change if there were more than three options? Has your opinion on who should win changed from the answer you listed in RE 1?

DQ 3) Try to think of a practical scenario where a group needs to make a decision, and argue which method is most suitable for that situation. What factors would change which method you choose?

Recommended Readings

Douglas, Andrew, et al. "Simulating Instant Runoff Flips Most Donald Trump Primary Victories." *FairVote*, 4 Mar. 2016.

For a fuller explanation of how Donald Trump's performance in the state primaries could have been greatly affected by a change in voting method, this source explains the methodology of the simulation and elaborates further on its findings.

Griffiths, Shawn M., et al. "Ranked Choice Voting Draws Historic Voter Turnout In 2018 Midterms." *IVN.us*, Foundation for Independent Voter Education, 8 Nov. 2018.

This article written for the Independent Voter Network examines the 2018 U.S. midterm elections, and looks at how a handful of cities recently switched from Plurality to Ranked Choice. The article largely focuses on the successes of these changes in terms of voter turnout, cost-effectiveness, and improved representation.

Hodge, Jonathan K., and Richard E. Klima. *The Mathematics of Voting and Elections: a Hands-on Approach*. Vol. 30, American Mathematical Society, 2018.

This book briefly introduces many of the same subjects in this chapter through a largely question-based approach. In general, these questions are formatted similarly to the Reading Exercises in this chapter, though tend to be a step up in difficulty.

Saari, Donald G. *Chaotic Elections!: a Mathematician Looks at Voting*. American Mathematical Society, 2001.

For a deeper, unbiased look at what we covered in this chapter, Chaotic Elections offers the most logical follow-up reading material. Saari explains a variety of voting methods (including all of the ones we covered) and unexpected ways they can impact elections. He reinforces these explanations with mathematical examples, proofs, and visuals that span a wide range of ability levels.

"Single Transferable Vote." *Electoral Reform Society*, 2017.

This article about Ranked Choice offers a brief summary of how the method works and what its merits are. It should be noted that some claims are disputable, and the source is somewhat biased in favor of the method. The Electoral Reform Society as a whole, however, offers well explained, brief descriptions of a wide variety of voting methods, and standardizes how they're compared to each other by assessing how well they meet specific indicators of representation.

Exercise Key:
Check Your Understanding 2)

Ranked Voter Preferences	
5	~~A~~ > B > C
1	~~A~~ > C > B
3	B > A > C
1	C > A > B
1	B > C > A
6	C > B > A
total 17	

Total votes for B? (B>C + B>A>C + B>C>A) = 5+3+1 = 9

Total votes for C? (C>B + C>A>B + C>B>A) = 1+1+6 = 8

We see that there were five people who thought A>B>C, and only one whose preference was A>C>B. So, with A no longer an option, five extra votes would go to B, and one more vote would go to C. Now, B ends up with nine and C ends up with eight—B wins!

RE 5)
Given a Ranked Choice election with the following results, we can immediately calculate the number of first place votes each candidate received in the first round:

Candidate	Votes (Round 1)	% Votes (Round 1)	Votes (Round 2)	% Votes (Round 2)
A	20	27%	0	0%
B	24	32%	34	45%
C	31	41%	41	55%

Information from Round 1:
Voters who rank A in first place? (A>B>C +or A>C>B) 20
Voters who rank B in first place? (B>A>C +or B>C>A) 24
Voters who rank C in first place? (C>B>A +or C>A>B) 31

Information from Round 2:
To calculate the number of voters who believed A>B>C vs A>C>B, we must look at where the votes were reallocated after Candidate A was dropped from the election. Since A had the fewest votes, the 20 voters who had originally listed it as first place then have their votes cast instead for their second choices. Therefore, we can see how many people had B or C listed as a second choice based on the difference in the number of votes they received in Round 1 vs Round 2.

Voters who ranked A in first place, and B in second? (A>B>C)
Total A>B>C voters = #B votes in Round 2 - #B votes in Round 1
= 34 - 24
= 10 voters

Voters who ranked A in first place, and C in second? (A>C>B)
Total A>C>B voters = #C votes in Round 2 - #C votes in Round 1
= 41 - 31
= 10 voters

Since neither B's votes nor C's votes had to be reallocated, we don't have enough information to find out the second and third preferences of any of the voters who listed B or C as first. Therefore, all the information we have is that 24 voters listed some ordering with B in first, and 31 listed an ordering with C in first.

Completed Table:

Ranked Voter Preferences	
10	A > B > C
10	A > C > B
24	B > C > A or B > A > C
31	C > B > A or C > A > B
75	

CHAPTER 2: APPORTIONMENT

There are 435 congressmen and congresswomen in United States House of Representatives (we refer to as the *House*). Each represents a portion of a state. For example, 36 of the 435 people represent the state of Texas. Every 10 years, the government must decide how many representatives each state should have based on the census. The Constitution as written states that "Representatives shall be apportioned among the several States which may be included within this Union, according to their respective Numbers." Where the Senate was founded on the idea of equal representation between states, the founders of this country created the House of Representatives with the idea of *proportional representation*: the larger a state, the more people they should have representing them in the House of Representatives. The problem of deciding how many people each state should get is known as the problem of apportionment.

However, the concept of apportionment is more than just a problem for the government. The same problem occurs for any limited resource that must be split between different groups. Say you have a pile of 100 books to give away to two different schools in the area. If each school is a different size, how do you divide the books between schools fairly? The larger schools should get more books. What happens if the books don't divide equally? If you have two schools, one with 515 students and one with 485, do you give 52 books to the larger and 48 to the smaller? 51 to the first and 49 to the second? Do you try to give each half of a book and give 51.5 to the first and 48.5 to the second? Giving a school one more or one less book may not seem important, but in the U.S. government, giving a state one more representative can matter tremendously. Apportionment uses methods that seek to divide a whole number of limited resources in the most fair way possible.

In the context of the government these limited resources are the *seats*. We refer to the number of representatives each state gets -- the total number of congressmen and congresswomen they have representing them in the House of Representatives -- as the number of seats each state gets. Let's say we have 10 seats to split between these four states below. The apportionment would look like this:

Figure 1

State	Population	Seats Distributed
Alabama	20	4
Colorado	10	2
Delaware	5	1
Florida	15	3

The total population of these four states is 50 people. So for these 50 people, there are 10 seats. We can say for every five people there is one seat. So, we give each state one seat for every five people it has. This is easiest to see in the state of Delaware which has exactly 5 people, meaning it should get 1 seat. Then, Colorado has twice the population of Delaware (10) so it should get twice the seats (2). Florida has three times as many people as Delaware so $15 \div 5 = 3$ seats. Lastly, Alabama has four times as many people so $20 \div 5 = 4$ seats.

While that example worked out nicely, that's because apportionment is quite straightforward when all the numbers divide evenly; the problems occur when the numbers are decimals or fractions. Consider a different example: suppose instead Alabama has 12 people, Colorado has 6 people, Delaware has 5 people, Florida has 7 people (shown in Figure 2). Above, we divided the total population of all states by the total number of seats. Doing so told us that each seat should correspond to 5 people. In this example, the total population is 30 and there are again 10 seats to distribute. Each seat should represent $30 \div 10 = 3$ people. For some states this works out nicely. Colorado has 6 people, so if each seat represents 3 people, they should get 2 seats. Delaware, on the other hand, should get $5 \div 3 = 1.67$ seats. However, a state can't elect 0.67 of a representative, so what do we do now? Determining how to work with these fractions and distribute seats is the focus of the area of apportionment. And in learning the history of the area, you'll see that the method of apportionment has changed many times, and you'll see the mathematics and politics causing those changes. By the end of this chapter, you will have found that the apportionment method in place can have a meaningful effect on the political landscape and even the person in the Oval Office.

Here's an overview of the different methods the U.S. has used for apportioning the House over time. Notice the number of different methods used -- each a change involving both mathematical and political debate!

1792: *Hamilton's Method passed by House; vetoed by Washington*
1792-1842: *Congress passes and uses Jefferson's method*
1842: *Webster's method used*
1850s-1890s: *Hamilton's and Webster's method used together*
1890s-1920s: *Webster's method used*
1929-1940: *Huntington-Hill and Webster method used together*
1940-present: *Huntington-Hill used*

Figure 2

Unlike the first example, this next one requires working with fractions. The country has a total population of 30 people and 10 seats to be distributed.

State	Population	Seats Deserved
Alabama	12	4
Colorado	6	2
Delaware	5	1.67
Florida	7	2.33

As noted above, working with fractions forces us to think of how to distribute our 10 seats. For the states that work out nicely -- Alabama and Colorado -- we give them their share of seats. For Delaware and Florida, 5 and 7 do not divide nicely by 3. The ``proportional representation" for these states would be 1.67 and 2.33 seats. We must distribute the 4 seats we have between them so we think about who deserves the seat more based on the math. Should we give Delaware and extra 0.33 of a seat and take the same from Florida, giving each 2 whole seats? Or should we give Florida and extra 0.67of a seat and take the same from Delaware, giving 3 and 1 whole seats? We must round one of them up and since Delaware's 1.67 is closer to 2, we give Delaware the extra seat. Therefore, as you can see in the table, each are given 2 whole seats.

Figure 3

State	Population	Seats Apportioned
Alabama	12	4
Colorado	6	2
Delaware	5	2
Florida	7	2

Hamilton's Method

The method we just made above is actually known as Hamilton's method and was used for 40 years. This method of distributing based on the fraction of the whole population was first introduced by the first Secretary of the Treasury, Alexander Hamilton, in 1792. Hamilton's method followed a natural logic in its design:

1. Compute the *quota*: the total population divided by the total number of seats in the House
2. Divide the populations by this quota. This gives a whole number and decimal remainder known as the *fractional seat share*.

3. Give states the base number of seats in its fractional seat share. (4.5 given 4 & 3.23 given 3)
4. Round up the fractional seat share of states one at a time beginning with the state with the largest decimal, then moving on to the state with the second largest decimal, and so on. Do this until you've distributed the correct number of seats.

In our second example, seen in Figure 2, the quota was 3 people per seat. From this we found the *fractional seat shares* to be 4 for Alabama, 2 for Colorado, 1.67 for Delaware, and 2.33 for Florida. Once giving Alabama and Colorado their respective seats, Delaware was given a single seat and Florida was given 2 seats. One seat was left to give to the state with the largest decimal, which was Delaware at 0.67. This left us with the apportionment shown in Figure 3.

RE 1) Try to follow the instructions above in apportioning the sample country below. The country has a total population of 40 people and 8 seats to be distributed.

State	Population	Seats Apportioned
Alabama	15	
Colorado	6	
Delaware	10	
Florida	9	

Alabama Paradox

While Hamilton's method seems natural, mathematics shows that it has unforeseen challenges. A paradox is a problem within the method that contradicts what the method tries to do. In our use, a paradox is where Hamilton's method leads to a division of seats that seems unfair. The example below will illustrate one such paradox of Hamilton's method.

RE 2) Below are the populations of states in a country and the fractional seat shares. The country has a total population of 14 people. Find Hamilton's apportionment for the states first under a seat total of 10 seats and second under a total of 11 seats. What do you notice?

State	Population	Fractional Seat Share	10 Seats Apportioned	Fractional Seat Share	11 Seats Apportioned
Delaware	6	4.286		4.714	
Colorado	6	4.286		4.714	
Alabama	2	1.429		1.571	

Adapted from "Apportionment Paradox"

Although the populations of states did not change when adding an extra seat, Delaware lost a seat in the House. This problem is known as the Alabama Paradox because in 1881 while considering apportionment, Congress discovered that by using the then current system, Alabama would be entitled to 8 representatives in a House with 299 members, but would receive only 7 representatives in a 300 member House. Without any change in population, Alabama would receive fewer representatives in a larger House.

Population Paradox

The Alabama paradox occurs where the total number of seats is changed, but what about when the seats are constant and the populations change? The population paradox illustrates another scenario where Hamilton's method is seemingly unfair.

RE 3) Below are the populations of fictional states now and 10 years later when the country must apportion again. The country's total population is 1,050 then jumps to 1,323. The House is set at 7 seats. Find the Hamilton's apportionment for these states in each decade. Do you notice anything that seems to be unfair?

State	Population	Fair Share	Seats Apportioned
Alabama	752	5.013	5
Colorado	101	0.673	1
Delaware	99	0.660	1
Florida	98	0.653	0

State	Population 10 Years Later	Fair Share	Seats Apportioned
Alabama	753	3.984	4
Colorado	377	1.995	2
Delaware	96	0.508	0
Florida	97	0.513	1

Adapted from Young's Equity: In Theory and Practice

In this example, Florida lost population while Alabama gained population, yet Alabama lost a seat and Florida gained one. This seems odd in that no state should be able to lose a seat while gaining population to a state that lost population. This can also happen when two states have populations increasing at different rates; a small state with large growth can lose a seat to a big state with smaller growth.

The Hamilton method was first proposed by Alexander Hamilton in 1792 but vetoed (refused) by President George Washington. Despite the Alabama and Population Paradoxes that can be seen mathematically, it wasn't either of these flaws that led to Washington's veto. Instead Washington vetoed Hamilton's plan because failed to satisfy the Constitution's requirement of at least 30,000 people per representative. Washington instead approved Thomas Jefferson's proposal (which we will explain below). In his method, Jefferson lowered the size of the House from 120 to 105 which was enough to ensure no state had a "persons to Representative" ratio of less than 30,000. Despite its many flaws, Hamilton's method was later reintroduced and used from 1842 to 1890.

Jefferson's Method

In Hamilton's method, we divided by the quota to get a proportional allocation of seats to each state called the fractional seat share. We then gave each state its base number but needed to round some fractional seat shares up to get the right number of seats. Jefferson's method, instead, divides by a different number -- the *divisor* -- similar to the quota. The difference being where the quota was chosen to be a specific number, Jefferson's method modifies its divisor. Specifically, Jefferson's method varies the divisor until it works out that you don't have to round any state!

This method was approved by Washington and was used to apportion the House of Representatives from 1792 to 1842. But its apportionment of the original 1792 House is almost identical to Hamilton's. If Hamilton's method had been enacted in 1792 on a House of size 105, 13 of the 15 states would have been assigned the same number of seats as they would receive under Jefferson's method. In the first instance of a change to apportionment, Jefferson used his method as a means to gain advantage for his own state of Virginia. The only two states that differed between the methods were Virginia and Delaware. Virginia gained a seat in the House under Jefferson's, whereas Delaware lost a seat of its own. While Hamilton's method falls victim to the Alabama and population paradoxes, Jefferson's method does not.

Jefferson's method works on the idea that you can vary the number you divide each population by in order to get an allocation for the number of total seats you desire. This number you vary is known as the *divisor*.

Figure 4

Say we want 10 seats split between the states below. First we set the divisor to be the quota and find the fractional seat share. Below, we start with that quota being 75. Next, Jefferson's method would tell us to round down.

State	Population	Fractional Seat Share	Rounded Shares
Florida	410	5.47	5
Kansas	260	3.47	3
Delaware	80	1.07	1

The problem is this divisor gave us 9 total seats between the three states when we wanted 10 total. Divisor methods allow us to change the number we divide by each time until we get a total number of seats we want. The modified divisor will always be lower than the quota so that there are no seats left over. The following is the same example but using the number 68 as the divisor.

Figure 5

State	Population	Fractional Seat Share	Rounded Shares
Florida	410	6.03	6
Kansas	260	3.82	3
Delaware	80	1.18	1

This is how Jefferson's method would be used on a large amount of states. This method allows us to ignore what happens to the fractions of seats and instead we round down and vary the divisor until we get a total seats we want. Where Jefferson rounded down, other methods call for us to round based on different rules, therefore giving different apportionments. These methods are: Jefferson's Method - always rounds down, John Adams' Method - always round up, Daniel Webster's Method - round to the closest integer, and finally the method we use today, Huntington-Hill which tells us to round from something called a geometric mean (we will go over this later).

One interesting way to translate Jefferson's method to simpler terms is to use a table. This is not obvious. You have not yet learned how to implement Jefferson's method using a table but this example can be referenced when interpreting the provided definition. The following is the Jefferson table for a country with 3 states with their respective populations and a total of 5 seats to allocate. Underlined values are attributed seats for their states.

Figure 6

State	Population	P÷1	P÷2	P÷3	P÷4	P÷5
A	200	**<u>200</u>**	**<u>100</u>**	**<u>66.7</u>**	50	40
B	100	**<u>100</u>**	50	33.3	25	20
C	100	**<u>100</u>**	50	33.3	25	20

The following method can be used to visually apportion based on Jefferson's apportionment as displayed above.

1. Create a table where each state can be compared. Create columns where the population of each state is divided by the number 1, and then a column with the population divided by 2, and then 3, and so on up to the number of total seats. Create rows for each state
2. Compare these numbers for each state. Circle the biggest number, then the second biggest, then the third biggest, and so on until you have circled as many numbers as there are seats (for example, to apportion 10 seats we circle the 10 biggest numbers)
3. Give a seat to each state where you circled a number in the row.

RE 4) Now, follow the same steps on an example of your own. This may be easier with a calculator. Complete the table below, determining apportionment for the states. Here there are 5 seats to distribute.

State	Population	P÷1	P÷2	P÷3	P÷4	P÷5
D	300					
E	100					
F	100					

Hamilton's v. Jefferson's

At this point you may be thinking, "Does it actually matter which we use?" The answer is a definite yes. Take, for example, a close presidential election. When a candidate wins the state, that person receives the number of seats that the state has in Congress as part of the Electoral College. If the number of seats a state has is changed then the candidate who wins receives a different amount of electoral college votes. The electoral college decides the winner by majority of the total seats, so the apportionment of states is fundamental to determining who wins.

RE 5) Take a fictional country made up of three states with a total population of 70 people. Each of the candidates is either from party D or party R. The size of the House of Representatives is set at 7 seats. Find the Jefferson and Hamilton apportionments for the country. Compare the effect on a presidential election between D and R. Who wins under each apportionment method discussed?

State	Population	Who won the state	Fractional seat share	Hamilton Apportionment	Jefferson Apportionment
A	34	D			
B	20	R			
C	16	R			

Table for Jefferson Apportionment:

State	P÷1	P÷2	P÷3	P÷4
A				
B				
C				

In this example, party R wins 4 electoral college votes to 3 under Hamilton's method and party D wins 4 electoral college votes to 3 under Jefferson's method. This shows the method in place can affect the Electoral College totals and even the person in the Oval Office. While Florida may have decided the 2000 election by less than 1,000 votes, the apportionment system can also have a significant effect on the winning candidate. It turns out that election using Hamilton's method would have elected George Bush while using Jefferson's method would have elected Al Gore.

Quota Rule

Hamilton's method isn't the only apportionment method that falls victim to paradoxes. All divisor methods violate something called the quota rule. The quota rule dictates that all states should receive either the fractional seat share rounded up or down. This translates to a scenario where a state deserving 4.33 seats to either receive 4 or 5 seats, but no higher or lower. The following example shows how Jefferson's method can violate this.

Figure 7

A country has three states. Each with populations listed below. The country has 100 seats to distribute. The quota is 9.99 to begin, but only gives us 98 seats so a modified divisor of 9.83 is used to apportion the 100 seats.

State	Population	Fractional Seat Share	Modified Seat Share	Apportionment
Alabama	98	9.810	9.969	9
Colorado	689	68.969	70.092	70
Delaware	212	21.221	21.567	21

Adapted from "Chapter 9: Apportionment"

This example shows us that a modified divisor, while still apportioning 100 seats correctly, can also give a state a number of seats that seems unfair. Colorado has a fractional seat share of 68.969 so it should receive either 68 or 69 seats. However, when the divisor is modified allowing for Jefferson's rounding down procedure, the new seat share for the state jumps to 70.092 leaving it with 70 seats. All methods that use modified divisors such as Jefferson's, Webster's, Adams', and the Huntington-Hill method can violate the quota rule.

Balinski-Young Impossibility Theorem

As we've found, Hamilton's method and its alternatives all display unfair distributions at times. This leads to apportionment's Impossibility theorem which states there is no apportionment system that has the following properties: it avoids violations of the quota rule, it does not have the Alabama paradox, and it does not have the population paradox (Balinski). This means that while we may strive to create a perfect apportionment method that is fair to states in all situations, one simply does not mathematically exist. So just as with voting theory, there can be no perfect figure (besides a single state) and we must employ imperfect methods at our own risk.

Apportionment Bias
As explained earlier, Jefferson's method favored his own state of Virginia in the apportionment of 1800 but the method actually affects much more than that. In general, Jefferson's method favors larger states due to his procedure of rounding down the fractional parts. For example:

Figure 8

State	Fair Share	Fractional Part Lost	Portion of Seats Lost
New York	30.5	0.5	1.6%
Delaware	1.5	0.5	33.3%

While each state lost the exact same fraction of a seat, the portion of the total seats lost of Delaware was far greater than the portion lost by the larger state of New York. Delaware lost 0.5 ÷ 1.5 or one-third of its seats when rounded down whereas New York lost 0.5 ÷ 30.5 or one-seventieth of its seats. Conversely, John Adam's method of always rounding up favors the smaller states much more.

Figure 9

State	Fair Share	Fractional Part Gained	Portion of Seats Gained
New York	30.5	0.5	1.6%
Delaware	1.5	0.5	33.3%

While each state gained the exact same portion of seats when rounded up, the portion of the total seats gained of Delaware was far greater than the portion gained by the larger state of New York. Thereby, smaller states benefited far greater under Adam's proposed plan.

Huntington-Hill
Due to the Alabama and Population paradoxes we discussed above, Congress stopped using Hamilton's method and transitioned to Webster's divisor method in 1900. Later, the 1920s saw Joseph Hill of the U.S. Census Bureau propose a new method which we still use today. His method strived to keep the "people per Representative" ratios of each state as equal to any other as possible. His first step was to choose the total number of seats, and then allocate seats to the states so that no transfer of a seat between states could make the quotas more equal. Edward Huntington, a former schoolmate of Hill's, saw his proposed method, and corrected an error in Hill's calculations thus attaching his own name. However, it wasn't for another 20 years that their method would be chosen by Congress.

In 1941, the apportionments computed by the Webster and Huntington-Hill methods differed in two states. Webster's assigned 18 seats to Michigan and 6 to Arkansas; Huntington-Hill assigned 17 to Michigan and 7 to Arkansas. At the time, Michigan was a state that leaned Republican whereas Arkansas leaned Democratic. Democrats controlled Congress, so they seized the mathematical opportunity to alter the apportionment system in their favor and gain another seat for their party. A Representative from Arkansas sponsored a bill to use Hill's method to apportion the House. Congress passed the bill and the Huntington-Hill method has been used to apportion seats in a 435 member House ever since. In the years following the change, the political tension over apportionment was out of control because the method by which we used was controlled by the ruling party in Congress. Thus, it was agreed that the method would not be up for discussion any longer and that the size of the House would not grow past 435.

As discussed earlier, the method we use in dealing with fractions, rounding up or down, has an effect on the mathematical bias of the apportionment. By changing the divisor in each column in the table we can adjust bias. In this same sense, Huntington-Hill rounds based on something called the geometric mean to minimize the bias. The concept of the geometric mean[4] is central to Huntington-Hill in that it translates the motivations of the authors into a practical method. This also makes this method harder to do by hand.

Figure 10

Using the same information as Exercise 3 in Jefferson's method, below is what the table looks like for Huntington-Hill. Notice how the column headers are now dependent on the geometric means of the seat numbers. The math behind how this table method is related to the goals written by Hill and Huntington is not obvious. The procedure for the table does not change, simply the numbers themselves. Underlined values are attributed seats for their states.

State	Population	P÷1.41	P÷2.45	P÷3.46	P÷4.47	P÷5.48
D	300	**212.77**	**122.45**	**86.71**	67.11	54.74
E	100	**70.92**	40.82	28.90	22.37	18.25
F	100	**70.92**	40.82	28.90	22.37	18.25

[4] The geometric mean is an average where two numbers are multiplied together, and then the square root is taken. The two numbers used are consecutive in this case. The geometric mean between 3 and 4 is $\sqrt{(3)(4)}=\sqrt{12}=3.46$

The Future of Apportionment

The discussion over how we determine how many Representatives each state should have continues to this day in the form of the following two proposed methods.

Fractional Voting Apportionment

Steven Salzburg, a professor at Johns Hopkins University, introduced fractional apportionment as a way to make Congress more responsive to its districts (Salzburg). The idea goes that both the winner and second place candidate in a district election would receive seats in the House of Representatives. However, each would receive a seat proportional to the percent of the total vote they receive. For example, in a district with 75 Republican voters and 25 Democratic voters, Salzburg would have the Republican candidate get 0.75 seats, while the Democratic candidate would get 0.25 seats. This would reflect the district's political preferences as there are triple the Republican voters and the Republican candidate gets triple the seats. As a whole, the district still gets one seat but every voter is represented. This congressional apportionment system requires that only the two candidates receiving the greatest vote share would receive fractional votes. These "winners" would preferably be achieved with ranked choice voting.

Salzburg also claims fractional voting apportionment has the ability to override gerrymandering (see Gerrymandering chapter for more details) and be more representative of the state's political position. He provides a contrived example with four Congressional districts in a state with a populus split 50-50 between the two major parties: Stars and Diamonds.

Figure 11

★	★	★	★	❖
★	★	❖	❖	❖
★	★	❖	❖	❖
★	★	❖	❖	❖

In this example, he supposes that the Democratic state party has drawn districts such that one single district is comprised of 80% Star voters and 20% Diamond voters. This would leave the three remaining districts with 60%-40% majorities for the Diamonds. Under the current system, the Diamonds would win three Congressional seats while the Stars would only win one. Despite the fact that the state is equally split between the two parties, the Diamonds have drawn districts in a way to win themselves a significant majority of seats in Congress. Under the fractional voting system, four Diamonds and four Stars would be elected to seats but with unequal individual voting weights. The Star candidate from the packed district would receive 0.80 votes and the rest of his party's winners would receive 0.40 Congressional votes. On the opposite side, the Diamond from the packed district would receive 0.20 Congressional votes and his compatriots 0.60 Congressional votes apiece. Thereby, each party receives four representatives and two total Congressional votes which falls in line with the voters and overcomes the gerrymandering in play (Salzburg).

RE 6) Each district below has a total population of 100,000 people. Calculate the party's seats divided by the total seats to find the party's representation in Congress. Do this for both fractional apportionment and our current single-winner system. Calculate the apportionment of seats to each party based on a state's information below.

Race	Democratic Votes	Democratic Fractional Vote	Republican Votes	Republican Fractional Vote
District 1	45,000		55,000	
District 2	60,000		40,000	
District 3	30,000		70,000	
District 4	45,000		55,000	
Totals:	180,000		220,000	

Thus in a general statewide view we see the effects of this alternate system:

Dem. Popular Vote	Single-winner Democratic Apportionment	Democratic Fractional Apportionment	Rep. Popular Vote	Single-winner Republican Apportionment	Republican Fractional Apportionment
45%	%	%	55%	%	%

Notice the difference between the single-winner and fractional apportionments. Thus, Salzberg's method follows exactly in line with popular vote thereby mitigating gerrymandering. Although this method makes great progress towards its goals, it also raises consequences associated with expanding the House to 870 members and using ranked choice voting.

Multi-member Districting

Where fractional voting elects two candidates to legislative seats, multi-member districting elects the same number of representatives from each state but does so in a very different way. Multi-member districts are those that elect multiple members to the House of Representatives. In states with less than five seats, representatives would be elected in statewide races. In states of more than five seats, representatives would be elected in districts of three to five members. In the state of Georgia pictured below, 14 districts currently exist but under the redistricting, districts could be drawn like those on the right with two districts (A & B) electing three members and two (C & D) electing four members.

Figure 12

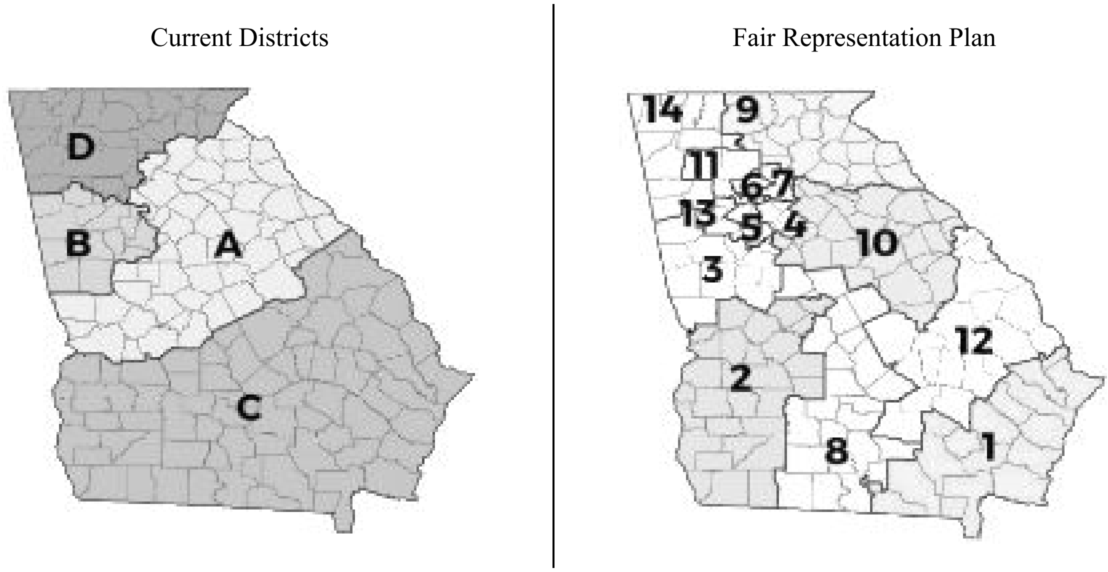


Figure from "The Impact of the Fair Representation Act: African American Voting Rights and Representation in the Deep South."

Where the current voting system uses plurality, this form of multi-member districting requires voters to select multiple winners so a form of ranked choice voting is implemented (see Voting Theory for more info). Each voter ranks candidates and for the first round, every vote cast counts for its top choice. If a candidate has more votes than the election threshold (explained below), they win, with any extra votes being distributed to help their next choices. If not all seats are filled, then the candidate with the fewest votes is eliminated, and votes cast for that candidate now count for the next choice instead. This process repeats until all seats are filled with candidates over the threshold.

The election threshold is the number of votes that guarantees a victory. For example, in an election between two candidates where there is only one winner, a majority or 50% + 1 vote is needed to win. If there are three candidates and one winner, the winner still needs the same majority but ranked choice voting would be used to eliminate a candidate, transferring their votes to the remaining candidates. In a multi-winner election, the threshold is less. For multi-member districts, it will be between 16.7% and 25% in every state with at least three seats. Say, for example, you have an election between four candidates and you need to select two winners. In order to guarantee that a candidate wins, each winner would need a third of the total votes or 33% + 1 vote. This new threshold is found by asking what percentage guarantees no other candidate can beat them. For any one-winner elections that is 100% ÷ 2 + 1 vote. For any two-winner election that is 100% ÷ 3 + 1 vote. This trend continues for any number of winners. The table for the relative thresholds is shown below. CVAP is the citizen voting age population.

Figure 13

Number of Seats in District	% of CVAP Needed to Elect Candidate
1	50% + 1
2	33.3% + 1
3	25% + 1
4	20% + 1
5	16.7% + 1

Adapted from "The Impact of the Fair Representation Act: African American Voting Rights and Representation in the Deep South."

A lower threshold is what allows minority voices to earn their fair share of representation on a reliable basis. The percentage of African American voters living in districts in the South where their CVAP meets the threshold to elect a representative would greatly increase, going from 40% to 98%(Richie). This would also allow minorities like Republicans in Massachusetts and Democrats in Georgia to elect representatives that share their political views. In the example districting maps of Georgia shown above, the currently districts yielded ten Republicans and four Democrats based on past election data, but the "Fair Representation Plan" or multi-member map would lead to somewhere from eight to nine Republicans and five to six Democrats. In other words, America isn't as politically divided as most people think; it simply seems that way because of winner-take-all elections portraying vast regions of the country as entirely red or blue.

Summary

The chronology of apportionment methods follows a narrative of political maneuvering, leading the country to its current method. As we've found, each method can have its own bias based on the way it distributes seats and each method falls victim to its own paradoxes. In comparing methods, we've also seen the apportionment method used can have a profound effect on national elections such as the presidency. Next, multiple new methods are being introduced and discussed in attempts to address what some believe are weaknesses of our current distribution systems. Finally, that distributing integer resources between groups can pose problems of fair division but due to paradoxes there can be no perfect solution.

Discussion Questions

DQ 1) Do you believe the method by which we apportion Congress should be changed? Why or why not?
DQ 2) Does the idea that your vote may be 'wasted' on a losing candidate affect you?
DQ 3) Envision the consequences of a multi-member or fractional vote system. How does it affect the political landscape as a whole?

Recommended Readings

"Apportionment Paradox." *Wikipedia*, Wikimedia Foundation, 23 Oct. 2018.

This article provides a cursory understanding of the various apportionment paradoxes explored above. Tables were adapted from this article and it can help in overviewing the history of apportionment.

Balinski, Michel L., and H. Peyton. Young. *Fair Representation: Meeting the Ideal of One Man, One Vote*. Brookings Institution, 2002.

Widely regarded as the "textbook" for studying the mathematics behind apportionment, this resource provides a thorough narrative of the government's usage of the different methods. While driven by calculus and high-level concepts, this resource can provide extension for topics explored in this chapter.

"Chapter 9: Apportionment." *www.coconino.edu*.

Written for a college classroom, this resource provides an accessible entry into Jefferson's method among others. This resource reviews the quota rule paradox and provides robust examples for interpreting violations.

Richie, Rob, et al. "The Impact of the Fair Representation Act: African American Voting Rights and Representation in the Deep South." *FairVote*, 20 June 2017.

The FairVote website is a modern thinktank looking at ways to improve the current apportionment method. This website also covers modern initiatives to address topics such as gerrymandering and electoral systems.

Salzberg, Steven. "The Problem With Our Democracy Isn't Gerrymandering. It's Integers." *Forbes*, Forbes Magazine, 12 Nov. 2018,

Written following the November 2018 election, this article introduces an alternative form of apportionment complete with examples and thoughtful debate over its feasibility. Less mathematics driven than other resources on this list, this article was written for a broad audience.

Young, H. Peyton. *Equity: in Theory and Practice*. Princeton University Press, 1995.

A deep dive into the statistics behind voting theory and apportionment, this resource is written with an understanding of high-level concepts. This resource complete with numerous tables and examples to portray inequity and drawbacks of many systems of government.

Exercise Key:

RE 1)

State	Population	Seats Apportioned
Alabama	15	3
Colorado	6	1
Delaware	10	2
Florida	9	2

RE 2)

State	Population	Fractional Seat Share	10 Seats Apportioned	Fractional Seat Share	11 Seats Apportioned
Delaware	6	4.286	4	4.714	5
Colorado	6	4.286	4	4.714	5
Alabama	2	1.429	2	1.571	1

RE 3)

State	Population	Fair Share	Seats Apportioned
Alabama	752	5.013	5
Colorado	101	0.673	1
Delaware	99	0.660	1
Florida	98	0.653	0

State	Population 10 Years Later	Fair Share	Seats Apportioned
Alabama	753	3.984	4
Colorado	377	1.995	2
Delaware	96	0.508	0
Florida	97	0.513	1

RE 4)

State	Population	P÷1	P÷2	P÷3	P÷4	P÷5
D	300	**300**	**150**	**100**	75	60
E	100	**100**	50	33.3	25	20
F	100	**100**	50	33.3	25	20

RE 5)

State	Population	Who won the state	Fractional seat share	Hamilton Apportionment	Jefferson Apportionment
A	34	D	3.4	3	4
B	20	R	2	2	2
C	16	R	1.6	2	1

Table for Jefferson Apportionment:

State	P÷1	P÷2	P÷3	P÷4
A	**34**	**17**	**11.3**	**8.5**
B	**20**	**10**	6.6	5
C	**16**	8	5.3	4

RE 6)

Race:	Democratic Votes	Democratic Fractional Vote	Republican Votes	Republican Fractional Vote
District 1	45,000	0.45	55,000	0.55
District 2	60,000	0.60	40,000	0.40
District 3	30,000	0.30	70,000	0.70
District 4	45,000	0.45	55,000	0.55
Totals:	180,000	1.80	220,000	2.20

Dem Popular Vote	Single-winner Democratic Apportionment	Democratic Fractional Apportionment	Rep Popular Vote	Single-winner Republican Apportionment	Republican Fractional Apportionment
45%	25%	45%	55%	75%	55%

CHAPTER 3: PARTISAN GERRYMANDERING

What is Partisan Gerrymandering?

In the United States, states get broken up into districts and each district elects its own officials. As we'll see, the districts can be drawn strategically to give a political party more power. This is known as partisan gerrymandering. In this chapter of *Winning With Math*, we'll examine how partisan gerrymandering is done, the effects it has, the tools used to combat it, and finally the future of districting in the United States.

Imagine a town with nine citizens who are voting to establish the town's new official name—Sunnyville or Cloudyville. If the town made its decision based off the popular vote, Figure 1a shows us that five people would vote for Sunnyville while only four would vote for Cloudyville. A majority prefers Sunnyville, so the town gets named Sunnyville.

Figure 1a

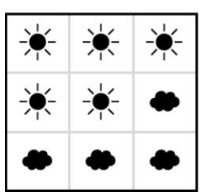


Figure 1b

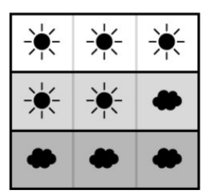
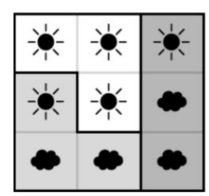


Figure 1c



Suppose instead that the town didn't want to make every citizen vote on this silly issue—the town is faced with many decisions and it would be difficult to have everyone vote on every issue all the time. Instead, the town breaks itself up into three equal districts. Each district will have three people, and those three people will vote for a single representative for their district. The representatives from each district then form a three-person town council which will vote on the naming issue. This plan of districting is very similar to how the House of Representatives in the United States works.

By breaking the town up into districts, the decision becomes more complicated than when it was left to a popular vote: different ways of drawing the districts could lead to different town councils, which might change what name the council picks. Examine the districts drawn in Figure 1b. Here, the top two districts have more people in favor of Sunnyville so each of them elect a candidate who favors Sunnyville. Meanwhile, the bottom district overwhelming prefers Cloudyville so they will elect a candidate who favors Cloudyville. In total, candidates favoring Sunnyville win two districts while candidates favoring Cloudyville win one. Therefore, when the town council votes, there will be more representatives favoring Sunnyville, and the town will be named as such.

On the other hand, with the districts drawn in Figure 1c, candidates favoring Sunnyville win one district while candidates favoring Cloudyville win two. When the town council votes, Cloudyville wins overall. Notice that without changing any individual's preferences, we completely flipped the outcome. This is the power of partisan gerrymandering.

In the United States, the House of Representatives uses a very similar form of districting as the town above. As we explored in the apportionment chapter, states get some number of representatives ("seats") in the U.S. House of Representatives based on their population. For example, if a state gets three seats, that state should be broken up into three districts. Each district should have roughly the same number of people and only the voters in each district get to elect the representative for that district. This is the same setting as Figures 1a-1c: a town of nine people got broken into three districts and each district elected a single representative to the town council.

Effects of Partisan Gerrymandering in the United States

Partisan gerrymandering influences political elections in the United States. Here are three examples when Democrats and Republicans have used partisan gerrymandering to their advantage.

During the 2012 election in Pennsylvania (Figure 2a), Democrats beat Republicans by approximately 50%–49% (the other 1% went to third party candidates). With these results, one could reasonably expect Democrats to win a majority of Pennsylvania's eighteen congressional seats. The actual result? Democrats won five congressional seats while Republicans won a whopping thirteen seats (72%) despite having a minority of the popular vote.

Figure 2a: 2012 House Elections Pennsylvania

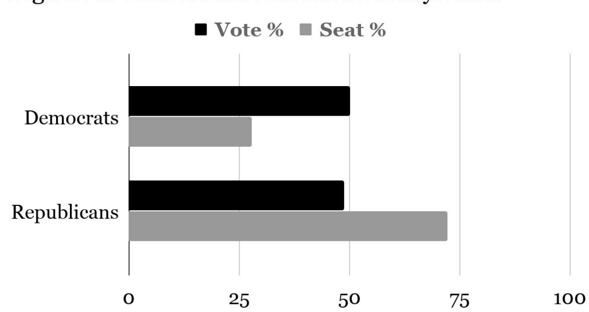
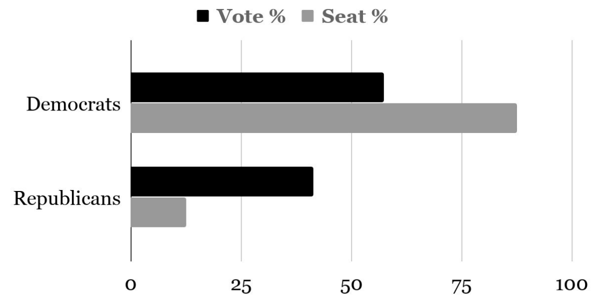
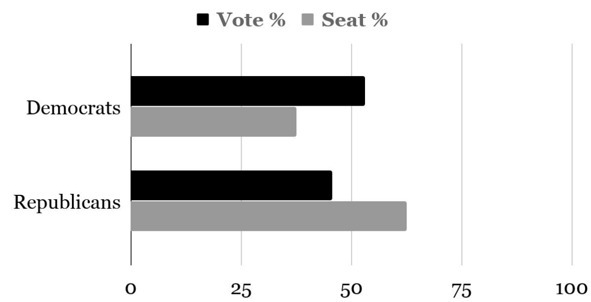


Figure 2b: 2014 House Elections Maryland



This is not an isolated incident. As seen in Figure 2b, in 2014 Democrats captured seven out of the eight congressional seats in Maryland with only around 57% of the vote. Republicans, who received approximately 41% of the vote, only won one congressional seat.

Figure 2c: 2018 House Elections Wisconsin



In Wisconsin during the 2018 election (Figure 2c), Democrats beat Republicans by approximately 53%–46%. Despite this, Republicans, who received a minority of the total vote, won five out of the eight congressional seats (63%).

Similar cases can be found in many states, during many election cycles, dating back as far as 1812. Note that there are many reasons for why seat and vote shares don't exactly match up. However, large gaps, especially when one party wins a majority of the seats with a minority of the votes, can suggest that districts might be drawn to advantage one party over another. Later in this chapter, we'll discuss several cutting-edge mathematical tools which help identify when partisan gerrymandering is at play.

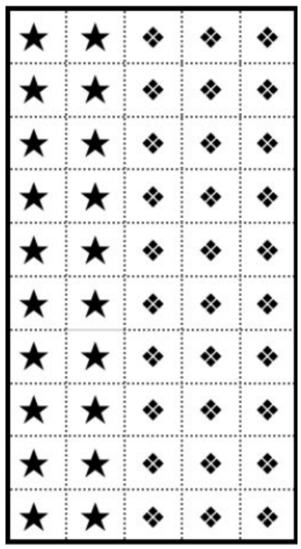
Squaretopia

Welcome to Squaretopia, an example we'll use to better understand the mathematics of gerrymandering! We'll first analyze the potential impact of gerrymandering.

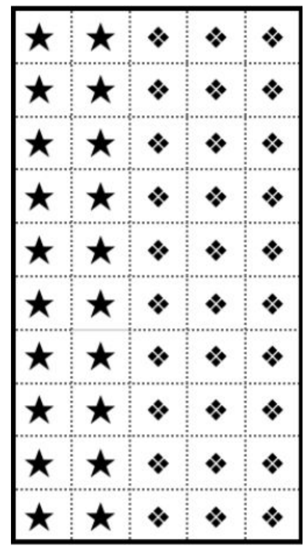
Squaretopia has a total population of fifty which must be split into five districts of equal size, containing ten people each. Additionally, each square in a district must be completely connected to the rest of the district. In this example, each square represents a person whose either prefers the *Star* party or the *Diamond* party.

RE 1a) You are asked to draw a new set of districts for Squaretopia. Create the boundaries for five evenly sized districts in a way that you believe will create the fairest outcome. (Hint: this is intentionally open-ended. There are no right or wrong answers.)

RE 1a



RE 1c



RE 1b) With the districts you just created, which party won? How many seats did they win?

Winning Party ________ Losing Party ________ Seats Won by Winning Party ________

RE 1c) You are now asked to draw a new set of districts for Squaretopia—this time with a twist. Try to get as many seats as possible for whichever party lost in the previous exercise. (Use the figure for RE 1c above.)

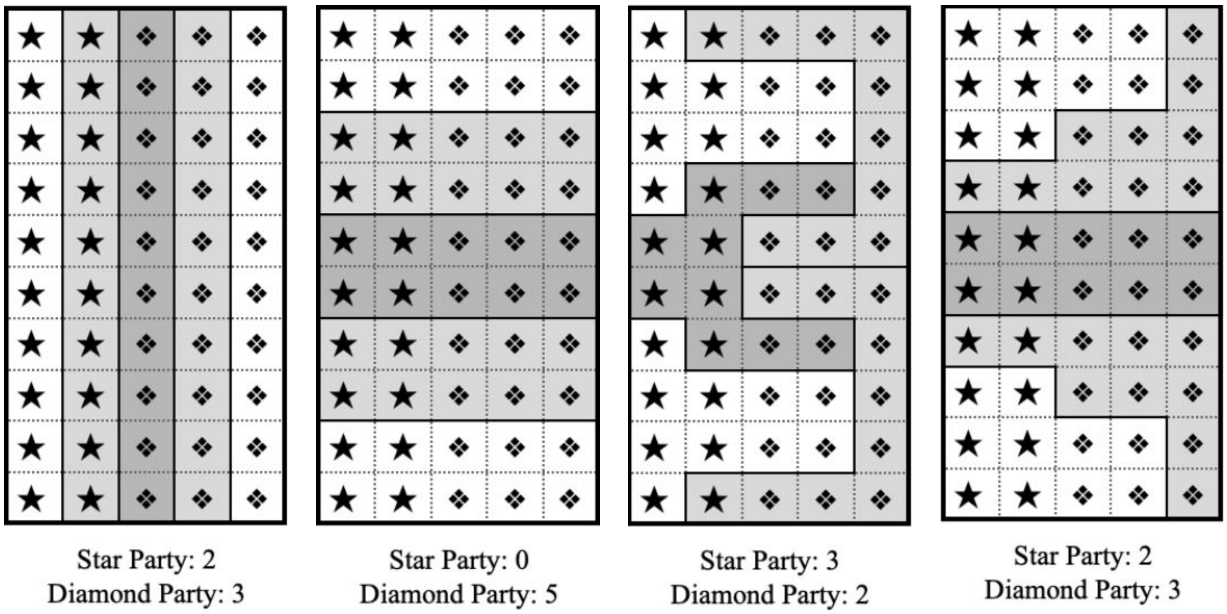
RE 1d) With the new districting plan from *RE 1c*, did you manage to flip the result of the election? If so, which party won this time? How many seats did they win?

Winning Party ________ Losing Party ________ Seats Won by Winning Party ________

The moral of the exercise is that creating districts can be challenging—there is no one perfect solution. Let's examine a few possible districts that could be created in Squaretopia to gain better insight into this idea.

Figure 3 shows four possible ways of breaking Squaretopia up into five districts. In Figure 3b, for example, every district has 4 Star voters and 6 Diamond votes. Since each district has a majority of Diamond votes, the Diamond Party wins all five districts.

Figure 3a **Figure 3b** **Figure 3c** **Figure 3d**



Reading exercise 1 is adapted from Stephen Nass

Examine the four districting plans above. All four plans are distinct and provide a wide range of outcomes, from Figure 3b which delivers a decisive 5–0 victory to the *Diamond* party, to Figure 3c which results in a slim 3–2 victory for the *Star* party. Regardless of these differences, there is no “right” or “perfect” districting plan. In fact, we’re about to see how each districting plan above is imperfect in some way.

On first glance, Figure 3a is a very intuitive districting plan. Of all the fifty votes cast, twenty were for the *Star* party and thirty were for the *Diamond* party. Therefore, it makes sense in this districting plan that two districts are won by the *Star* party and three districts are won by the *Diamond* party. However, this districting plan is not perfect. First, the districts are long and narrow, stretching all across Squaretopia. In other words, the districts are not very “compact.” We’ll discuss compactness in more detail later, but ideally voters in the same district should live fairly close to each other and not be at opposite ends of the map. As a general rule of thumb, more compact districts are preferred as they often do a better job at representing local communities. Another concern with this districting plan is how it leads to very uncompetitive elections. Each district in the map is won in a landslide victory because each district either goes 100% for the *Star* party or 100% for the *Diamond* party. Districting plans that produce more competitive elections are generally favored by voters, as each individual vote will have a greater impact on the outcome of an election.

The districting plan in Figure 3b fixes both the problems discussed with the plan from Figure 3a. The districts are both more compact and more competitive (there is now a 60%–40% split in each district). However, this districting plan does not reflect the popular vote. Even though 40% of voters support the *Star* party, it wins zero districts in the election. Likewise, the *Diamond* party is overrepresented, winning all five districts while only maintaining 60% of popular support.

While Figure 3b is biased towards the *Diamond* party, Figure 3c is biased towards the *Star* party. In this case, the *Star* party wins a majority of the districts with a minority of the votes. The districts in this plan are carefully crafted to maximize the *Star* party's chances of winning. This is done using two strategies known as "packing" and "cracking"—we'll learn about these in the next section.

Finally, Figure 3d is an attempt to create a "fair" districting plan. As you can see, it does a pretty good job: the districts are relatively compact, competitive, and the plan's outcome reflects that of the popular vote. With that being said, many of the districts are oddly shaped and spread out. In other words, you can't make everyone happy, and this districting plan is not an exception.

Packing and Cracking

Packing

The strategy of "packing" is simple but effective. To pack a district, you attempt to dilute support for your opponent by creating several districts where they win by an extremely large amount. These are often districts you know your party won't win anyway, so you are essentially admitting defeat in them with the intention of wasting as many votes for the other party as possible (that is, "packing" them into the already-lost districts). If, for example, you drew a district where the opposing party won with a 9–1 margin (as in Figure 3c of Squaretopia), you're effectively wasting three of the opposing party's votes. This is because in order to win, the opposing party just needs six votes. Any votes above six have no real impact on the election so if your opponent had not been packed, these votes could have helped them win (and you lose) other districts.

Cracking

The goal of "cracking" is to spread out the remaining votes for the opposing party across many districts. To do this, you create districts where your party can safely but narrowly win—you want just enough votes to confidently win a district, but not enough where you're "wasting" your support. An example of this is seen in Figure 3c, where the three districts won by the Star party are won on a narrow 6–4 basis. This maximizes the impact each Star voter has on the election, while simultaneously "wasting" votes for the Diamond party: in this case, the four votes that went to the Diamond party were wasted because the district was lost. If used elsewhere, these votes might have helped *Diamond* voters win another district.

When creating a gerrymandered districting plan, the strategies of "packing" and "cracking" are often used together. The goal is to "pack" your opponent into a few districts where they win by an extremely large amount and then "crack" their remaining support among the rest of the districts in order to make it extremely difficult for them to win.

The Origins Of Gerrymandering

The term "gerrymandering" originates from the districting plan created by Elbridge Gerry in 1812. Mr. Gerry, governor of Massachusetts at the time, agreed to a districting plan that created very lopsided districts to avoid areas of Federalist support and favor the Democratic-Republican party. (The Federalists and Democratic-Republicans were the two major parties at the time.) The resulting districting plan can be seen in Figure 4. During a dinner party with many prominent Federalists in March of 1812, an artist compared one district to a snake-like monster, another guest commented that it looked like a Salamander. Eventually the term "gerrymander" was coined.

Figure 4

Gerrymandering Then vs. Now

Some Good News

Most districting plans are created after the US Census, which only happens every ten years. Much can occur over those ten years—voter preferences change, demographics shift, etc. For a long time, these changes couldn't be predicted very well. This meant that after just one or two election cycles, the effect of most gerrymandering attempts disappeared. Therefore, while the gerrymandered districts created by Mr. Gerry initially had their intended effect, they didn't hold up well over time: they only gave a small, temporary advantage to Mr. Gerry's party.

Now the Bad News

In the past few decades, with the advent of technology, the districting process has become much more sophisticated and gerrymandered districts hold up significantly better to changes over time. In part, this is because mapmakers now have easy access to voter information, demographic data, and online databases containing masses of information: likes and dislikes on social media, online purchases, licenses, registrations, magazine subscriptions, and more. Expert mapmakers then use this information, historical election results, and algorithms that predict demographic and voter preference changes to form accurate predictions of future voting patterns. These predictions are analyzed and overlaid on a map of a region in a sophisticated mapping software where, with a few clicks, mapmakers can craft districts to their liking.

The Silver Lining

While technology has made gerrymandering more effective than ever, it has also led to the development of new tools and techniques which help detect and combat gerrymandering. In the next three sections, we'll explore some of the most common mathematical techniques to detect gerrymandering including geometric formulas, the efficiency gap, and partisan symmetry. When working with some of these techniques, you may find it useful to use a calculator if you have access to one. If not, no worries: we'll include any computations needed at the end of this chapter.

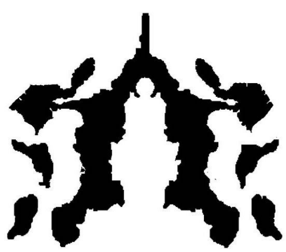
Geometric Methods of Detecting Gerrymandering

One of the classic ways of detecting gerrymandering is based on the idea that, if a district looks weird, it might be because of partisan gerrymandering. In this section, we'll see how math helps us measure how weird-looking a district is. We'll start, though, with examples.

The Rorschach Test

The Rorschach test is a psychological test where participants examine a series of ten cards. Each card has an ambiguous inkblot image. Participants are then asked to describe what they see.

RE 2) This is our own version of the Rorschach Test—with a twist. Two of the images below are cards from the actual test and two of them are a gerrymandered district that is repeated (so the left half would be the actual district and right half would be it's reflection). Can you identify which is which?



2a) Inkblot / District

2b) Inkblot / District

2c) Inkblot / District

2d) Inkblot / District

Rather tricky, right? At the end of this section, we'll reveal the correct answers and what the actual gerrymandered districts are.

We briefly mentioned in our Squaretopia example that compactness is generally favored in districting as compact districts often do better at representing local communities. But what does "compactness" really mean? While there is general agreement that districts should be reasonably compact, a big question is how we mathematically measure this. Two potential options are the Polsby-Popper and Reock scores. Let's investigate both to see how they work.

Polsby-Popper Score

The Polsby-Popper score compares a district's area (the size of a district) to the square of its perimeter (the total length of a district's border). An uncompact district will have a low score close to zero while more compact districts will have higher scores. For a little intuition, often times intentionally gerrymandered districts have weird shapes with large perimeters. Written mathematically, the formula for the Polsby-Popper score looks like this[5]:

[5] Generally, the district's area in this formula is multiplied by 4π, so that the score range falls between zero and one. In our case, we will leave this out in order to simplify calculations.

$$\frac{District's\ Area}{(District's\ Perimeter)^2} = \frac{District's\ Area}{District's\ Perimeter \times District's\ Perimeter}$$

Figure 5a and 5b each show one district that is enclosed inside a circle. ***For now, don't worry about the circle when calculating the Polsby-Popper score****. The important part is the district in Figure 5a is a* 1×9 *rectangle, and the district in Figure 5b is a* 3×3 *square.*

Figure 5a

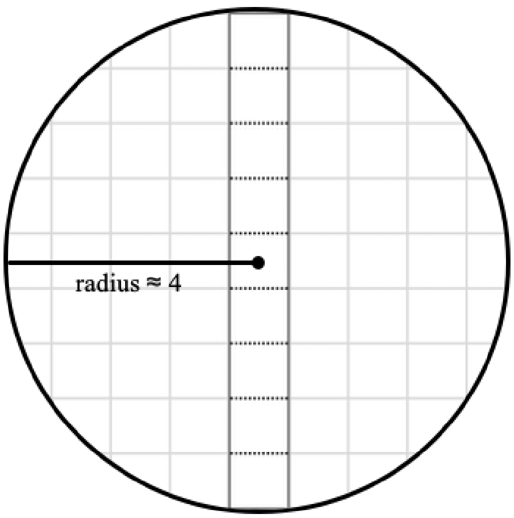
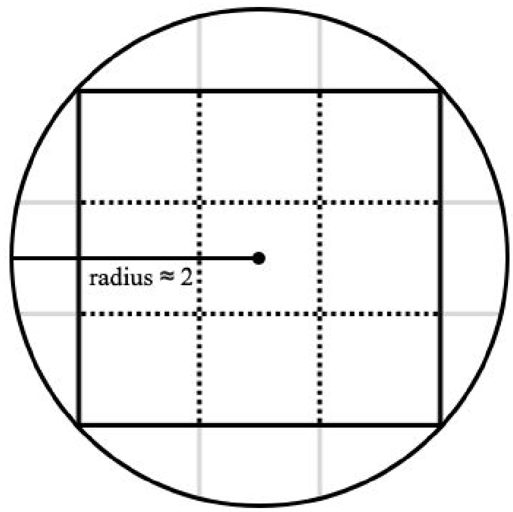


Figure 5b



Example: Calculate the Polsby-Popper score for the district in Figure 5a

$District's\ Area = 9$

$District's\ Perimeter = 1 + 9 + 1 + 9 = 20$

$(District's\ Perimeter)^2 = 20 \times 20 = 400$

Polsby-Popper score $= \frac{District's\ Area}{(District's\ Perimeter)^2} = \frac{9}{400} = 0.0225$

0.0225 is really close to zero, suggesting that the district is uncompact. Looking at the Figure 5a, this makes sense as we see that voters are about as far apart from each other as possible.

RE 3a) Do you think the district in Figure 5b will have a higher or lower Polsby-Popper score than the district in Figure 5a? Why?

RE 3b) Calculate the Polsby-Popper score of the district in Figure 5b. (If you don't have access to a calculator, just leave your answer as a fraction). Does the result agree with what you predicted above?

Reock Score

The Reock score is the ratio of a district's area to the area of the smallest possible circle that fully surrounds the district. Similar to the Polsby-Popper Score, an uncompact district will have a low score while more compact districts will have higher scores. For a little intuition, districts where people are very spread out require large circles to enclose them while compact districts don't. Written mathematically, the formula for the Reock score looks like this[6]:

$$\frac{District's\ Area}{Area\ of\ Enclosing\ Circle} \approx \frac{District's\ Area}{3 \times Radius\ of\ Enclosing\ Circle \times Radius\ of\ Enclosing\ Circle}$$

RE 3c) Before we calculate the Reock Score for Figure 5a and Figure 5b, make a prediction. Which district will have a higher Reock Score? Why?

<u>Example</u>: Calculating the Reock score of the district in Figure 5a

$$District's\ Area\ =\ 9$$

$$Area\ of\ Enclosing\ Circle\ =\ 3\ \times\ 4\ \times\ 4\ =\ 48$$

Reock Score $= \frac{District's\ Area}{Area\ of\ Enclosing\ Circle} = \frac{9}{48} = 0.1875$

RE 3d) Calculate the Reock Score for the district in Figure 5b. (If you don't have access to a calculator, just leave your answer as a fraction). Does the result agree with what you predicted above?

[6] Generally, the area of the enclosing circle is calculated using the formula πr^2, and the score range would fall between zero and one. In our case, we will approximate π as 3 in order to simplify calculations.

The Polsby-Popper and Reock score are great tools which help detect partisan gerrymandering but we cannot depend on them completely. While oddly shaped districts are *often* gerrymandered to help one party, not all oddly shaped districts *are* gerrymandered. An example of this is district four in Illinois (Figure 6a), which resembles a pair of earmuffs. While not compact, experts in the field contend that district four is not gerrymandered as it follows the traditional districting principle of keeping communities intact: the odd shape serves to connect two Hispanic population centers.

Figure 6a
Illinois District 4 (2018)

Figure 6b
Indiana State Map (2018)

Conversely, compact districts aren't necessarily unbiased. The districts in Indiana (Figure 6b) are notably compact, however they often lead to very skewed election results. Under this districting plan, Republicans won seven out of nine seats with approximately 55% of the total vote in 2018. Therefore, while compactness is a useful indicator for gerrymandering, we cannot blindly rely on it without understanding the broader context of a districting plan.

As promised, here are the results to our Rorschach test from the beginning of this section: RE 2a and RE 2d are taken straight from the actual Rorschach test. RE 2b and RE 2c correspond to Maryland's third (Figure 6c) and Tennessee's second district (Figure 6d). Theses two districts have a Polsby-Popper score of .003 and .009 respectively, using our formula from above. In this case, it makes sense that Maryland's third district has a score closer to zero, as the shape of it is significantly "weirder" than that of Tennessee's second district. Meanwhile, Indiana's first district (top-left district in Figure 6b), has a Polsby-Popper score of .045, significantly higher than the other two. This should make sense as the shape of the district (roughly rectangular), it is by far the most compact of the three.

Figure 6c
Maryland District 3 (2018)

Figure 6d
Tennessee District 2 (2018)

The Efficiency Gap

In 2015, Nicholas Stephanopoulous and Eric McGhee developed a promising mathematical formula for standardizing whether or not a district is gerrymandered. This formula, called the efficiency gap, has recently been at the center of many major court cases on gerrymandering. It is calculated by comparing the number of votes "wasted" by each party in an election. Remember from our Squaretopia example where some districts were won with a 9–1 margin and others with just a 6–4 margin? This difference is part of what the efficiency gap detects!

There are two types of wasted votes:

1. Votes for the *winning party* in a particular district beyond the 50% majority needed to win.
2. All the votes for the *losing party* in a particular district.

Votes are wasted in the first case because once a party receives over 50% of the votes in a given district, they are guaranteed to win. The difference between having 51% or 100% of the votes doesn't matter: in either case, the party will get a representative for the entire district. Instead, votes cast in excess of 50% would be better used to give that party an advantage in another district.

For the second case, if a party loses, the difference between having 1% or 49% of the vote doesn't matter. Since none of the votes for the losing party in a district help elect a candidate anyway, they would be better used in districts where they could help other candidates from their party win.
<u>Example</u>: Calculating the wasted votes of each party in District 1 in Figure 7a.

Figure 7a

District 1	❖	❖	❖	★	★	★	★	★	★	★

In District 1, the *Diamond* party receives 3 total votes and the *Star* party receives 7. Therefore, the *Star* party wins the district, and the wasted votes for the *Star* party will be those cast in excess of 50%. The total votes for the *Star* party (7) minus the number needed to win (5)[7] equals the *Star* party's total number of wasted votes (2). Since the *Diamond* party was the losing party, all their votes will be considered wasted (3). So, for District 1, there are 2 wasted votes for the *Star* party and 3 wasted votes for the *Diamond* party.

The efficiency gap compares the total number of wasted votes by each party with the logic that, in a fair districting plan, each party should waste roughly the same number of votes. This ensures no party is being targeted to waste more votes than the other. The formula for the efficiency gap is as follows:

$$\text{Efficiency gap} = \frac{\#\, Wasted\ Votes\ Party\ 1 - \#\, Wasted\ Votes\ Party\ 2}{\#\, Total\ Votes\ From\ Both\ Parties}$$

Note that values we plug in for (# Wasted Votes Party 1) and (# Wasted Votes Party 1) are the total number of wasted votes across an entire districting plan, not a single district. When the results of the efficiency gap are between ***-0.08*** and ***0.08***, a districting plan is considered fair. If the value falls outside of this range, however, Nicholas Stephanopoulous and Eric McGhee suggest that gerrymandering could have occured. Do keep in mind however, that no one metric will be a 100% perfect measure for whether or not gerrymandering has taken place.

Figure 7b below represents a town with four districts where each district has ten people who prefer either the *Star* Party or the *Diamond* Party. Let's calculate the efficiency gap of this town to see if the districting plan systematically favors one party over the other.

Figure 7b

District 1	❖	❖	❖	★	★	★	★	★	★	★
District 2	❖	❖	❖	★	★	★	★	★	★	★
District 3	❖	❖	❖	❖	★	★	★	★	★	★
District 4	❖	❖	❖	❖	❖	❖	❖	❖	❖	❖

[7] While it would technically take 6 votes to win in this district (in order to prevent a tie), the efficiency gap counts the votes cast in excess of 50%. Therefore 10 X 0.5 = 5 votes are needed to win for our calculations.

RE 4a) Complete the chart below with the correct values. The first row has already been filled in with the values from our previous example (Figure 7a).

	# Diamond Votes	**# Star Votes**	**# Wasted Diamond Votes**	**# Wasted Star Votes**
District 1	3	7	3	2
District 2				
District 3				
District 4				
Total	20	20		

RE 4b) Calculate the efficiency gap of the districting plan in Figure 7b above. (Hint: recall the efficiency gap formula from above and remember to use your calculated ***total*** values from *RE 4a*.)

RE 4c) Based off the value you calculated for the efficiency gap, do you believe is plan is gerrymandered? Why or why not?

Applications

The efficiency gap has been used in the major Supreme Court case *Gill v. Whitford* to support the claim that the 2011 districting plan in Wisconsin was gerrymandered. Researchers found that the efficiency gap was approximately .13 during the 2012 election and .11 during 2014. For context, every election and respective districting plan between 1972 and 2000 in Wisconsin had an efficiency gap of approximately zero. The sharp increase in the efficiency gap between then and now led William Whitford, the plaintiff of the case, to suspect intentional partisan gerrymandering.

Concerns
While the efficiency gap is a great tool for detecting gerrymandering, it has limitations. For example, the efficiency gap favors districting plans that create 25%–75% splits: these make an equal number of votes be wasted by either party (losing party wastes 25% of the vote, winning party wastes $75\% - 50\% = 25\%$ of the vote). Therefore, forcing states to have districting plans with a low efficiency gap might not be ideal, as they might try to have lots of 25%–75% districts. Doing so would yield very uncompetitive elections, making elected officials less accountable to the public.

Additionally, there are concerns that the efficiency gap takes the incredibly complicated topic of districting and overly simplifies it into a formula that yields just a single value. Anytime such a complex issue is evaluated by only one number, we risk losing insight on the other important consequences of a decision. Therefore, any time someone promises you a single number to use to make all your decisions, it's worth being skeptical.

Partisan Symmetry

The mathematics of gerrymandering is modern and developing quickly. In this section, we'll discuss one more tool used to help catch and prevent gerrymandering. This is one of the most technical and mathematically involved tools, so if necessary, feel free to skip this section.

Another effective tool for detecting gerrymandering is called partisan symmetry. Partisan symmetry examines the results of an election under different voting outcomes. The idea is that if party A wins 70% of the seats with 60% of the vote, then party B should also win 70% of the seats in a scenario where they have 60% of the vote. In other words, partisan symmetry evaluates the fairness of an election by how much the results are subject to change based on different vote ratios.

To examine partisan symmetry, we construct a “seats per vote” graph[8]. Figures 8a and 8b are examples of partisan symmetry graphs from a Republican point of view. The percentage of votes a party receives (in this case Republicans) is measured on the horizontal, and the percentage of seats they earn is measured on the vertical. At the bottom left corner of Figure 8a, the point (0.0, 0.0) means that if Republicans receive 0% of the vote, they will earn 0% of the seats. In this case, we’ll also know that if Democrats receive 100% of the vote, they will earn 100% of the seats. Similarly, the point (0.6, 0.75) means that if Republicans receive 60% of the vote, they will earn 75% of the seats. In this case, we’ll also know that if Democrats receive 40% ($100\% - 60\%$) of the vote, they will earn 25% ($100\% - 75\%$) of the seats.

[8] Constructing these graphs is a little complicated. If you’re interested, we assume that there is uniform partisan swing in each district: if a party does better and is liked by 10% more of the voters nationally, we assume that it's liked by 10% more of the voters in each district. However, individual districts often vary about 5% from national preferences. Therefore, we simulate hundreds of elections, using different random local effects for each district (in order to account for the 5% variability), to create the final seats per vote graph.

In Figure 8a, Minnesota has an ***unbiased*** districting plan. Its center at roughly (0.5, 0.5) means that when both parties receive 50% of the vote, they are each win about 50% of the seats. Additionally, the graph is roughly symmetrical elsewhere, so any given percentage of the vote will yield roughly the same number of seats to either party.

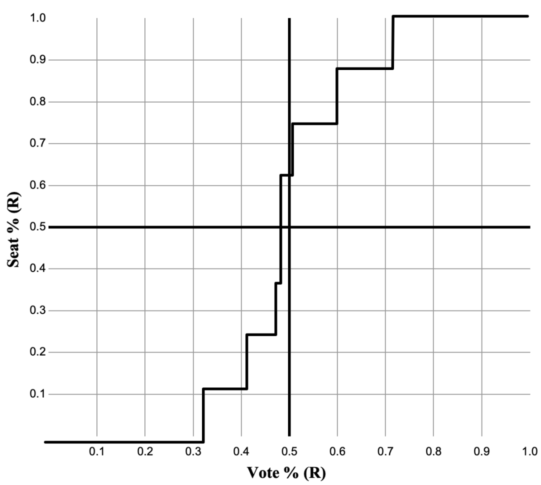


Figure 8a
Minnesota 2016

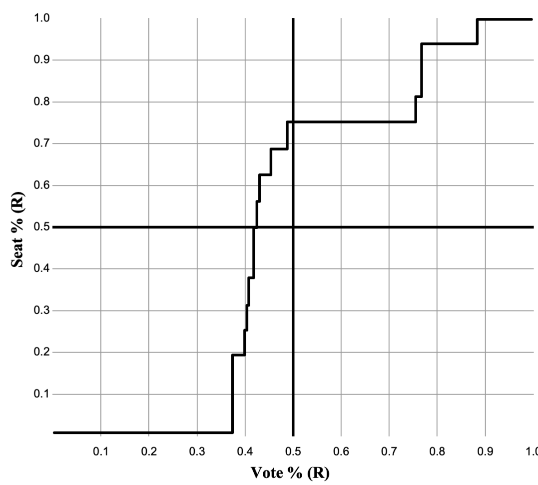


Figure 8b
Ohio 2016

Often, however, this is not the case. In Figure 8b, Ohio has a ***biased*** districting plan. This partisan symmetry graph contains the point (0.48, 0.76) which means that Republicans in Ohio could win 76% of the seats with just 48% of the vote. To find how many seats Democrats would earn with 48% of the vote, find the point (.52, .76). In that case, Republicans receive 76% of the seats with 52% of the vote. Therefore, Democrats would receive 24% (100% − 76%) of the seats with 48% (100% − 52%) of the vote. A result like this—one with a high level of asymmetry—is a good indicator that a districting plan may be biased.

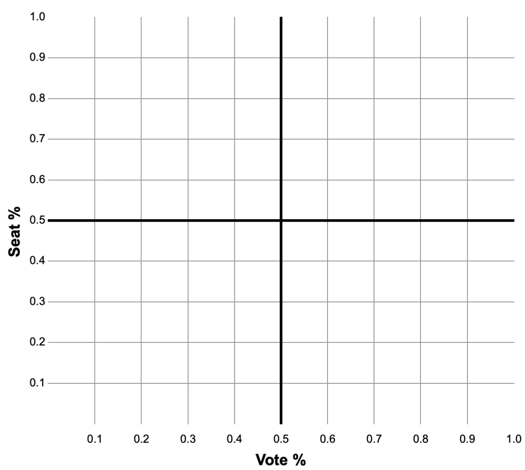
Partisan symmetry graphs also contain information about the nature of the districts in a districting plan. If a party gains substantially less than 1% of the seats with an additional 1% of the vote (so that the curve is very *flat*), then the districting plan is categorized as ***unresponsive***. Essentially, more votes don't significantly change the outcome of an election. This situation often arises when districts are gerrymandered to protect incumbents (the current holder of a particular office), and it's often viewed as harmful to democracy because it creates many "safe" seats. Candidates in safe seats are largely unconcerned with losing to the opposing party and may push further into their own ideologies to block off other candidates from the same party, potentially leading to a more polarized political climate.

If a party gains approximately 1% of the seats with an additional 1% of the vote, the plan is roughly ***proportional***. Essentially, more votes change the results of the election by just as much as one would expect. This seems like a logical and fair outcome in many people's eyes, however, it very rarely occurs due to the "winner takes all" structure of most US elections (the winner of a district, even through a very narrow margin, ends up representing the *entire* district).

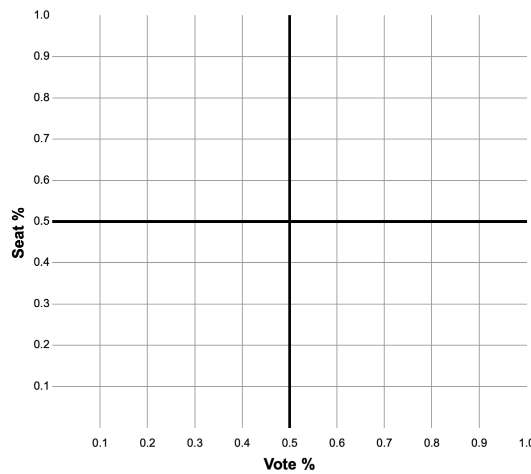
The “winner takes all” system means that very frequently, a party gains more than 2% of the seats with an additional 1% of the vote (so that the curve is very *steep*, as in Figure 8a). This type of districting plan is classified as ***responsive***. Essentially, each vote has a great impact on the results of an election. Often in a responsive districting plan, if a party wins, they will win by a large margin (like winning 65% of the seats with 51% of the vote). While this outcome is often confused for gerrymandering, it is not an issue so long as the graph is symmetrical and a similar outcome would occur if the opposing party won instead.

RE 5) Sketch what you think an unbiased-unresponsive and biased-responsive partisan symmetry graph would look like. (Hint: there are many ways to do this)

5a) Unbiased-unresponsive



5b) Biased-responsive



As with all of the tools we’ve examined, skewed partisan symmetry by itself is not enough to prove gerrymandering, as there could be other factors that influence the partisan symmetry of a districting plan. Nonetheless, these graphs are useful for flagging potential gerrymandering and examining the responsiveness of districts.

Other Explanations for Biased Partisan Symmetry

Many disagree that gerrymandering is the main cause of the bias seen in partisan symmetry graphs. Another reason could be that people with similar political affiliations tend to live in the same types of places. For example, Democrats receive a large amount of support in cities and majority-minority districts (districts where one or many minority groups make up the majority of the population). Because Democrats often win these districts by overwhelming margins, they end up wasting more votes than Republicans. Note that while Republicans typically receive more support in suburban and rural areas, the gap is not to the same level as that for Democrats in cities and majority-minority districts. Therefore, Democrats tend to win fewer districts with higher margins which could lead to naturally biased partisan symmetry graphs. However, even in difficult cases, creating districting plans with unbiased partisan symmetry it's very rarely impossible.

For example, New Jersey and Illinois both contain very Democratic cities and Republican suburban and rural areas. Additionally, they both have a high number of majority-minority districts that tend to lean Democratic. Even with these factors, both states still produce districting plans with roughly unbiased partisan symmetry graphs. Frequently, this comes at the cost of compactness, as we saw earlier when examining district four in Illinois (Figure 6a). However, the point is that creating districting plans with roughly unbiased partisan symmetry graphs is often possible.

Alternative methods of districting

The tools we've discussed—measures of compactness, the efficiency gap, and partisan symmetry—help identify and flag potential gerrymandering. As we've seen, however, these all have strengths and weaknesses. Nonetheless, the idea is that these tools (along with many others) together can build a strong argument for the courts that gerrymandering has occurred in a particular case.

Unfortunately, this is a reactionary approach that can only realistically be used to defeat some of the most extremely gerrymandered districting plans (as those are the ones you can build the strongest case against).

The usual way of creating districting plans (in a partisan state governments), however, is not the only way. Arizona, Iowa, California, Washington, Idaho, and New Jersey all use commission-based models instead. These commissions have varying levels of success, but they often minimize gerrymandering and lead to districting plans with less partisan bias.

California's Citizens Districting Commission was established between 2008 and 2010. Before the commission was created, the state government often drew districts with the intent of protecting incumbent candidates (the current holder of a particular office). This changed with the creation of the new Citizens Commission, made up of fourteen members: five Democrats, five Republicans, and four independents who serve on ten year terms. For a districting plan to be approved, three members from each group must support it. The first districting plan was created in time for the 2012 election, and the results were immediate—nine Congressional seats became competitive that weren't before. The California's Citizens districting Commission has been widely praised, and activists are pushing for other states to adopt similar models.

Summary

In this chapter, we first explored what partisan gerrymandering is and the effect it can have on election results: we've seen that the choices we make about drawing districts can completely swing elections. Then, we discovered the tactics of "packing" and "cracking," which have been used to gerrymander districts since 1812, when Elbridge Gerry approved the now-famous salamander-shaped district. Over time, technology's development has made gerrymandering more effective than ever, but strategies to identify and combat it have also advanced. Together, we explored powerful tools including measures of compactness, the efficiency gap, and partisan symmetry, which are often used in major court cases like *Gill v. Whitford* as evidence of gerrymandering. There is no magic solution to gerrymandering, but we've learned that we can fight some of the most extreme cases through mathematics and attempt to limit gerrymandering through the establishment of independent districting commission.

Karl Rove, a long time political consultant and policy advisor once said, "when you draw the lines, you make the rules." Know who's drawing the lines, because while a group or party may be winning with popular support behind them, it could also be that they're *winning with math*, when they really shouldn't be winning at all.

Discussion Questions

DQ 1) Is this your first time hearing about gerrymandering? Do you think anything should be done to prevent it? If so, what?

DQ 2) What tools to identify gerrymandering do you find most interesting or useful? Why?

DQ 3) If you were to write a letter on this subject to your congressman or congresswoman, what would you say?

Recommended Readings

Daley, David. Ratf*Ked: the True Story behind the Secret Plan to Steal America's Democracy. Liveright Publishing Corporation, A Division of W.W. Norton & Company, 2017.

This is probably the most interesting and accessible read if you want to continue exploring the subject of gerrymandering. This book is more partisan than the other recommended readings, however it presents the causes and effects of gerrymandering in an incredibly engaging way .

Duchin, Moon. "Gerrymandering Metrics: How To Measure? What's the Baseline?" Proceedings of Redistricting and Representation, American Academy of Arts and Sciences.

This paper covers the tools we used to detect gerrymandering in more depth and introduces new ones such as random map sampling. You'll find the content more technical than what we've covered, however after this chapter you have a great foundation if you want to explore this paper.

McGann, Anthony J., et al. Gerrymandering in America: the House of Representatives, the Supreme Court, and the Future of Popular Sovereignty. Cambridge University Press, 2016.

Gerrymandering in America is a great read if you want to learn more about the legal perspective on partisan gerrymandering and the many court cases that have lead the country to where it is today. Additionally, it goes into great depth on partisan symmetry, if you're curious to learn more about that.

Reading Exercise Key:

RE 1) Answers will vary. Figures 3a-3d show possible district maps that could have been used.

RE 2)

2.1 Inkblot	2.2 District	2.3 District	2.4 Inkblot

RE 3a) Compact districts have higher Polsby-Popper scores than less compact districts. Examining Figure 5b, we see that the district is more compact than the district in Figure 5a. Therefore, we would expect that the district in Figure 5b has a higher Polsby-Popper score than the district in Figure 5a.

RE 3b)

$District's\ Area = 9$

$District's\ Perimeter = 3 + 3 + 3 + 3 = 12$

$(District's\ Perimeter)^2 = 12 \times 12 = 144$

Polsby-Popper score $= \frac{District's\ Area}{(District's\ Perimeter)^2} = \frac{9}{144} = 0.0625$

0.0625 > 0.0225, which is expected as the district in Figure 5b is more compact than the district in Figure 5a and more compact districts have higher Polsby-Popper scores than less compact districts.

RE 3c)

Similar to the Polsby-Popper Score, an uncompact district will have a low score while more compact districts will have higher scores. Since the district in Figure 5b is more compact than the district in Figure 5a, we expect the district in Figure 5b to have a higher Reock score than the district in Figure 5a.

RE 3d)

$District's\ Area = 9$

$Area\ of\ Enclosing\ Circle = 3 \times 2 \times 2 = 12$

Reock Score $= \frac{District's\ Area}{Area\ of\ Enclosing\ Circle} = \frac{9}{12} = 0.75$

0.75 > 0.1875 which is expected as the district in Figure 5b is more compact than the district in Figure 5a and more compact districts have higher Reock scores than less compact districts.

RE 4a)

	# Diamond Votes	# Star Votes	# Wasted Diamond Votes	# Wasted Star Votes
District 1	3	7	3	2
District 2	3	7	3	2
District 3	4	6	4	1
District 4	10	0	5	0
Total	20	20	15	5

RE 4b)

Efficiency gap $= \frac{\#\ Wasted\ Votes\ Diamond\ Party - \#\ Wasted\ Votes\ Star\ Party}{\#\ Total\ Votes\ From\ Both\ Parties} = \frac{15-5}{40} = \frac{10}{40} = 0.25$

Note: -0.25 is also correct.

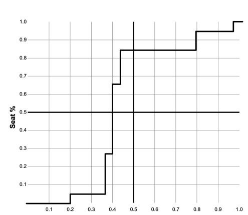
RE 4c)

This plan is probably gerrymandered because the calculated value for the efficiency gap (either 0.25 or -0.25) , is outside the -0.08 to 0.08 range for a fair districting plan.

RE 5)

Answers will vary. Sample Answers shown below

RE 5.1)



RE 5.2)